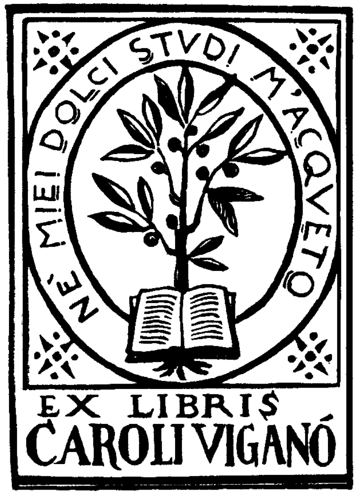






# ANALYSEOS SUBLIMIORIS

## OPUSCULA

*AUCTORE*

GREGORIO FONTANA

DE CLER. REG. SCHOLAR. PIARUM

PHILOSOPHIÆ ET MATHESEOS

PROFESSORE.

*Academiae Scientiarum Bononiensis Instituti Socio.*

VENETIIS, MDCCLXIII.

Typis SIMONIS OCCHI.

*SUPERIORUM FACULTATE, AC PRIVILEGIO.*

*CRESCENTINO BAVIERA*

EX MARCHIONIBUS MONTIS ALTI

*IN PROVINCIA ASTENSI*

EQUITI NOBILISSIMO

GREGORIUS FONTANA

Cler. Reg. Scholarum Piarum Felicitatem.

*Liceat mihi, Marchio Ornatissime,* TE *iisdem verbis compellare, quibus summus ac præcipuus totius Europæ Philosophus Alembertius Virum ingenii gloria præclarum, & Generis antiquitate spectatum Augustinum Lomellinum, qui nuper in Januensi Re-*

*Republica clavum tenebat*, *alloquebatur*; *nihil est enim*, *quod ad rem dici possit opportunius*, *quodque animi mei sensa vividius valeat exprimere*, *mentemque significantius declarare* : Les plus grands génies de l'Antiquité (*inquit ille*), mettoient le nom de leurs amis à la tête de leurs ouvrages, parce qu'un ami leur étoit plus cher qu'un protecteur. Un sentiment si digne de vous, est tout ce que je puis imiter d'eux. Ce n'est point à votre naissance que je rends hommage, ce seroit mettre vos Ancêtres à votre place, & oublier que j'écris a un Philosophe. L'accueil que vous faites aux Gens de Lettres ne leur laisse point appercevoir la supériorité de votre rang, parce que vous n'avez à leur envier la supériorité des lumieres. Aussi, non content de rechercher leur commerce, vous leur temoignez encore cette consideration réelle sur la quelle ils ne se méprennent pas quand ils en sont dignes, & comme la vanité n'a point de part à votre estime pour eux, la reputation ne vous en impose point dans vos jugemens (*a*). Silentio itaque prætteribo generosam & splendidam Generis Tui nobilitatem, & præclara Majorum Tuorum facinora, quæ cæteroquin tam amplam atque uberem dicendi segetem suppeditarent, ut haud facile esset orationis exitum invenire. Quis enim adhuc memoria non recolat FRANCISCUM MARIAM Præ-

(*a*) Recherches sur la Précession des Equinoxes par M. D'Alembert à M. le Marquis Lomellini.



*Præsulem amplissimum ex Equitibus Ss. Mauritii & Lazari*, *qui flagrantissimo in Patriam amore æterna Posteris & nunquam interitura reliquit munificentiæ suæ monumenta? Quis* JOANN. JOSEPHUM *Seniorem non admiretur*, *attonitusque suspiciat*, *qui quum sapientissime pluribus functus esset amplissimisque muneribus*, *magnamque sui expectationem omnibus reliquisset*, *abdicatis repente Ecclesiasticis Dignitatibus*, *animo ab ambitione invicto*, *nec vanæ gloriolæ splendore obcæcato optime & Patriæ*, *& Generi suo consuluit*, *ne nobilissimæ Familiæ germen interiret? Quis* MARCUM ANTONIUM, JOANN. JACOBUM *meritis valeat laudibus cumulare? Namque virtus apices quodammodo suos attigerit necesse est*, *ubi in sui admirationem vel ipsos potest adducere viros Principes. Nonne Franciscus Dux de Oratore ad Senatum Venetum* (*hunc autem cum nomino*, *Majestatis & amplitudinis Domicilium cujuscumque ob oculos obversatur*) *eligendo quum ageretur*, *in* MARCUM ANTONIUM *conjecit oculos? Hujus Urbis*, *& Arcis custodia quanti tunc temporis esset*, *novit quisquis in ejusdem Fastis non sit omnino hospes & peregrinus. Hæc tamen nonnisi alteri demandata est. Quam vero Hic sibi comparavit laudem amore*, *æquitate*, *prudentia*, *cæterisque dignis Moderatore dotibus*, *quantum per Deos immortales auxit*, *amplificavit* JOANN. JOSEPHUS *Junior Præsul & ipse ornatissimus*, *non in una aut altera*, *sed in pluribus Pontificiæ Ditionis Urbibus! Etenim gravissimis sui regiminis quamvis distentus curis*, *non secus*

*secus ac si cogitationes suas omnes, ac vitam ipsam litteris devovisset, summa veteris historiæ utilitate innumera e situ & squalore revocavit antiquitatis monumenta, iisque in Tuæ Domus vestibulo constitutis commonstratum omnibus esse voluit, bonarum Artium Cultoribus ipsam diu noctuque patefieri. Plura alia, splendidissimaque possem commemorare nobilitatis insignia, nisi compertum haberem, me ad Philosophum scribere, qui*

*Et genus, & proavos, & quæ non fecimus ipsi*
*Vix ea nostra vocat (a)*

*frequentibusque interdiu usurpare solet sermonibus auream illam Juvenalis sententiam*

*...... Miserum est aliorum incumbere famæ,*
*Ne collapsa ruant subductis tecta columnis (b).*

*Enimvero quum nobilitas nihil aliud sit, quam cognita virtus, quis in eo, quem veterascentem videat ad gloriam, generis antiquitatem desideret? (c) At silentio non premam quæ tua sunt, acre ingenium & peracutum, vividam mentis aciem, mirabilemque memoriæ celeritatem, quibus eo jam progressus es, ut de abstrusioribus philosophicis disceptationibus possis & ornate dicere, & copiose disserere, & perspicue, illuminate, distincte pronunciare sententiam. Namque illud in Te singulare prorsus est, & eximium, ut quæ in Philosophicis præsertim Scientiis cæteri summa animi contentione, impigro labore, diuturnis studiis vix, aut ne vix quidem adsequuntur, Tu ea veluti per ludum jocumque & subtilissime excutias, & explanes acutissime,*

(a) Ovid. Metam. XIII. (b) Juven. Sat. III. (c) Cic. ad Hirtium



*sime, nihil ut demum videatur tam abditum, alteque positum, quo acumen Tuum non possit eniti. Testes hujus rei sunt locupletissimi & qui Tecum jamdiu familiariter versantur, & ego in primis, quem non ultimo habes loco inter Tuos; mihi enim auspicato prorsus contigit, ut Tecum, quem antea suspiciebam & venerabar e longinquo, modo conjunctissime vivam. Quid vero de politiori Litteratura Poetice præsertim dicam, in qua tam mirifice excellis, ut Musarum quasi in gremio educatus videare? Extant Poemata Tua latina, & vernacula, quæ Horatiano deducta filo, & ad nobiliorum Poetarum lucernam elucubrata Veneres omnes, Gratias, & Lepores redolent, & divite quadam sententiarum verborumque copia aurei fluminis instar exuberant. Hæc ubi e scriniis, quibus eadem premis, in lucem exibunt, a bonarum artium Cultoribus quanto expectata cupidius, tanto avidius suscipienda, in aperto ponent, & quantus sis, & quanta polliceri sibi abs Te possint elegantiores litteræ, si vacuo, ut soles, tranquilloque animo pergas eisdem vacare. Quid vero de morum suavitate dicam, qua Tibi omnium animos adeo devinxisti, ut nihil Tua consuetudine antiquius habeant, cumque in omnium oculis habites, omnium delicium esse videaris? Quid de liberalitate & munificentia prope singulari, qua calamitosos homines miseriis, & egestate afflictos tam benevole sublevas, nihilque negas petentibus, immo hortaris ut petant? Quis Te cum vitæ suavitate severior? Quis cum ingenii venustate incorruptior? Quicquid Tibi superest, aut ex domesticis negotiis, aut ex litte-*

*litterarum ſtudiis, quæ prima ſemper apud Te fuerunt, id totum concedis temporibus amicorum. Sed nolo in laudes Tuas liberius exſpatiari, quum laudatore non egeat qui omnium meretur encomia. Enimvero quum eſſe, quam videri bonus ſemper malueris, quod de M. Catone teſtatum reliquit Salluſtius, nihilque recte feceris, ut feciſſe videreris, a laudum Tuarum ſermonibus mirifice abhorres; atque hinc fit, ut quo minus gloriam petis, eo magis adſequaris (a). Quod ſupereſt, illud eſt, ut Te obſecrem, ne hoc qualecumque munuſculum alio vultu ſuſcipias, quam quo ſoles omnia; meque ea foveas humanitate, qua Tua virtus omnis condita eſt. Hoc igitur, quodcumque eſt, æqui bonique conſule. Me Tibi totum, librumque meum trado & offero, & eo quidem animo, ut Tua virtute, non fortuna commovear.*

(a) Sallus. De Bello Catilin. §.LVII.



# DE FORMULARUM
## QUARUMDAM TRIGONOMETRICARUM INTEGRATIONE

### OPUSCULUM I.

Cum Regiæ Berolinenſis Academiæ Commentarios paucis ante diebus volutarem, incidi in eximiam Diſſertationem immortalis Leonardi Euleri inſcriptam: *Recherches plus exactes ſur l'effet des Moulins à Vent.* tom. XII. *Hiſt. De l'Acad. Roy. des Scienc. & Bell. Lettr.* année 1756. In hac Diſſertatione infraſcriptæ formulæ ita integratæ efferuntur. Eſt $\omega$ arcus radio 1 deſcripti.

$$1 \quad S\frac{d\omega}{\cos\omega} = L\,\text{tang}\,(45° + \tfrac{1}{2}\omega)$$

$$2 \quad S\frac{d\omega}{\cos\omega^3} = \frac{\sin\omega}{2\cos\omega^2} + \tfrac{1}{2}\,L\,\text{tang}\,(45° + \tfrac{1}{2}\omega)$$

$$3 \quad S\frac{d\omega}{\sin\omega^2\cos\omega} = L\,\text{tang}\,(45° + \tfrac{1}{2}\omega) - \frac{1}{\sin\omega}$$

$$4 \quad S\frac{d\omega\cos\omega}{\sin\omega^4} = -\frac{1}{3\sin\omega^3}$$

$$5 \quad S\frac{d\omega\cos\omega^3}{\sin\omega^6} = -\frac{1}{5\sin\omega^5} + \frac{1}{3\sin\omega^3}$$

$$6 \quad S\frac{d\omega}{\sin\omega} = L\,\text{tang}\,\tfrac{1}{2}\omega$$

7 $\int \frac{d\omega}{\sin\omega^{m}} = \frac{m-2}{m-1} \int \frac{d\omega}{\sin\omega^{m-2}} - \frac{\cos\omega}{(m-1) \times \sin\omega^{m-1}}$

Harum formularum integratione prius absoluta, proposui mihi Formulas quatuor omnium maxime catholicas, quæ alias omnes complecterentur, nimirum

I. $\int \frac{d\omega \sin\omega^{n}}{\cos\omega^{m}}$

II. $\int \frac{d\omega \cos\omega^{n}}{\sin\omega^{m}}$

III. $\int d\omega \sin\omega^{n} \cos\omega^{m}$

IV. $\int \frac{d\omega}{\sin\omega^{n} \cos\omega^{m}}$, in quibus exponentes $m$, $n$ sunt numeri positivi, & negativi, integri, & fracti, atque etiam nihilum, seu $o$. Sed antequam ad istarum Formularum integrationem accedamus, præstabit Eulerianas formulas exstricare, ut multiplices viæ, & methodi ad eandem veritatem perveniendi elucescant. Esto itaque

### LEMMA I.

Sint bini arcus radio 1 descripti $\phi$, $\theta$; erit tang $(\phi + \theta) = \frac{\tang\phi + \tang\theta}{1 - \tang\phi \tang\theta}$. Demonstratio passim invenitur apud Analystas.

Co-



Coroll. Sit $\phi = \theta$, eritque tang $2\phi = \frac{2 \tang\phi}{1 - \tang\phi^{2}}$.

### LEMMA II.

Sit tangens arcus simpli $= a$, arcus subdupli tangens $= x$; erit $x = \frac{-1 + \sqrt{a^{2}+1}}{a}$

DEM.

Ex Coroll. Lemm. I. habetur $a = \frac{2x}{1-x^{2}}$; ergo $x^{2} + \frac{2x}{a} - 1 = o$; unde eruitur $x = \frac{-1 + \sqrt{a^{2}+1}}{a}$ Q. E. D.

### LEMMA III.

$$\int \frac{dz}{(1-z^{2})^{p}} = \frac{z}{(2p-2) \times (1-z^{2})^{p-1}} + \frac{2p-3}{2p-2} \int \frac{dz}{(1-z^{2})^{p-1}}$$

Exponens p est integer, vel fractus, vel compositus ex integro, & fracto, &c.

DEM.

Fiat $\int \frac{dz}{(1-z^{2})^{p}} = \frac{M}{(1-z^{2})^{p-1}} + C \int \frac{dz}{(1-z^{2})^{p-1}}$, in qua æquatione M est functio ipsius z postea determinanda, C quantitas constans invenienda. Facta differentiatione erit $\frac{dz}{(1-z^{2})^{p}} = \frac{dM \times (1-z^{2}) + (2p-2) \times Mzdz +}{(1-z^{2})^{p}}$ $\frac{C\,dz}{(1-z^{2})^{p-1}}$, unde eruitur $dz =$

$-z^2 dM + dM + \overline{2p-2} \times Mzdz - Cz^2dz + Cdz$. Supponatur modo $M = Dz^n + Ez^{n-1} \ldots + Hz^2 + Az + B$; & differentiando erit $dM = nDz^{n-1}dz + \overline{n-1} \times Ez^{n-2}dz \ldots + 2Hzdz + Adz$; factaque ſubſtitutione valorum M, & dM in æquatione ſuperiori, ordinatiſque terminis, & nihilo æquatis, prodit

$$-nDz^{n+1} - \overline{n-1} \times Ez^n \ldots - 2Hz^3 - Az^2 + \overline{2p-2} \times Bz + A + \overline{2p-2} \times Dz^{n+1} + \overline{2p-2} \times Ez^n + nDz^{n-1} \ldots + \overline{2p-2} \times Hz^3 + \overline{2p-2} \times Az^2 + 2Hz + C = 0$$
$$-Cz^2$$
$$-1$$

In hac æquatione inſtituta de more coefficientium ſimilium comparatione, inveniuntur D, E ...., H uſque ad A nihilo æquales. Prodit vero $A = \frac{1}{2p-2}$, $C = \frac{2p-3}{2p-2}$, $B = 0$; ergo $M = \frac{z}{2p-2}$; proindeque

$$S\frac{dz}{\overline{1-z^2}^{p}} = \frac{z}{2p-2 \times \overline{1-z^2}^{p-1}} + \frac{2p-3}{2p-2} S\frac{dz}{\overline{1-z^2}^{p-1}}$$

Q. E. D.

His porro in anteceſſum conſtitutis Eulerianas formulas ad integrationem ita revocamus. Eſto prima $S\frac{d\omega}{\cos\omega}$. Sit x coſinus arcus $\omega$, eritque $S\frac{d\omega}{\cos\omega} = S\frac{-dx}{x\sqrt{1-x^2}}$; ad quam integrandam fiat de more $\sqrt{1-x^2} = u$; & invenietur $-xdx = udu$, & $-\frac{xdx}{x^2} = -\frac{dx}{x} = \frac{udu}{1-u^2}$, ac denique $\frac{-dx}{x\sqrt{1-x^2}} =$

du



$\frac{du}{1-u^2}$. Hinc $S\frac{-dx}{x\sqrt{1-x^2}} = S\frac{d\omega}{\cos\omega} = L\frac{\sqrt{1+u}}{\sqrt{1-u}}$; ac loco ipſius u, $\sqrt{1-x^2}$ ſubrogando, prodit $S\frac{d\omega}{\cos\omega} = L\frac{\sqrt{1+\sqrt{1-x^2}}}{\sqrt{1-\sqrt{1-x^2}}} = L\frac{1+\sqrt{1-x^2}}{x}$. Jam quoniam x eſt coſinus arcus $\omega$, erit $\sqrt{1-x^2}$ ſinus ejuſdem arcus, & $\frac{\sqrt{1-x^2}}{x}$ ipſius arcus tangens; proindeque tangens arcus dimidii, ſeu $\frac{1}{2}\omega$ (Lemm. II.) prodibit $= \frac{1-x}{\sqrt{1-x^2}}$. Hinc quoniam arcus 45° tangens = 1, erit (Lemm. I) tangens $(45^\circ + \frac{1}{2}\omega) = \dfrac{1 + \frac{1-x}{\sqrt{1-x^2}}}{1 - \frac{1+x}{\sqrt{1-x^2}}} = \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}} = \frac{1+\sqrt{1-x^2}}{x}$; ergo $S\frac{d\omega}{\cos\omega} = L\frac{1+\sqrt{1-x^2}}{x} = L \tang (45^\circ + \frac{1}{2}\omega)$ Q. E. I.

Scholium I. Nulla hic additur conſtans, quia evaneſcente $\omega$ prodit $L \tang (45^\circ + \frac{1}{2}\omega) = L \tang 45^\circ = L\, 1 = 0$.

Eſto nunc formula altera $S\frac{d\omega}{\cos\omega^3}$ Sumpta ut ſupra x pro coſinu arcus $\omega$ erit $S\frac{d\omega}{\cos\omega^3} = S\frac{-dx}{x^3\sqrt{1-x^2}}$; factaque $\sqrt{1-x^2} = u$, obtinebitur $S\frac{-dx}{x^3\sqrt{1-x^2}} = S\frac{du}{\overline{1-u^2}^2}$. Eſt autem (Lemm. III.)

A 3 $S\,du$

$\int \frac{du}{(1-u^2)^2} = \frac{u}{2(1-u^2)} + \frac{1}{2}\int \frac{du}{1-u^2} = \frac{\sqrt{1-x^2}}{2x^2} + \frac{1}{2}\int \frac{-dx}{x\sqrt{1-x^2}} = \frac{\sin\omega}{2\cos\omega^2} + \frac{1}{2} L\, tang\, (45^\circ + \frac{1}{2}\omega)$; ergo $\int \frac{d\omega}{\cos\omega^3} = \frac{\sin\omega}{2\cos\omega^2} + \frac{1}{2} L\, tang\, (45^\circ + \frac{1}{2}\omega)$. Q. E. I.

Schol. II. Nulla addenda est constans, quia evanescente $\omega$ fit $\frac{\sin\omega}{2\cos\omega^2} + \frac{1}{2} L\, tang\, (45^\circ + \frac{1}{2}\omega) = 0$.

Esto formula tertia $\int \frac{d\omega}{\sin\omega^2 \cos\omega}$ Factis iisdem ac supra subrogationibus erit hæc $= \int \frac{-dx}{x(1-x^2)^{\frac{3}{2}}} = \int \frac{du}{u^2(1-u^2)} = \int \frac{du}{1-u^2} - \frac{1}{u} = \int \frac{-dx}{x\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = L\, tang\, (45^\circ + \frac{1}{2}\omega) - \frac{1}{\sin\omega} + \varphi$. Q. E. I

Schol. III. Constans $\varphi$ invenietur supponendo $\omega = 0$, & prodibit $\varphi = \infty$ adeoque $\int \frac{d\omega}{\sin\omega^2 \cos\omega} = \infty$.

Supponimus enim tales esse hujusmodi formulas, ut evanescente arcu & ipsæ evanescant, quod interim adnotetur.

Esto formula quarta $\int \frac{d\omega \cos\omega}{\sin\omega^4}$; eritque =





$\int \frac{-x\,dx}{(1-x^2)^{\frac{5}{2}}} = \int \frac{du}{u^4} = -\frac{1}{3u^3} = -\frac{1}{3(1-x^2)^{\frac{3}{2}}} = -\frac{1}{3\sin\omega^3}$; ergo $\int \frac{d\omega \cos\omega}{\sin\omega^4} = -\frac{1}{\sin\omega^3} + \varphi$. Q. E. I.

Schol. IV. Hic quoque prodit $\varphi = \infty$; adeoque $\int \frac{d\omega \cos\omega}{\sin\omega^4} = \infty$.

Esto formula quinta $\int \frac{d\omega \cos\omega^3}{\sin\omega^6}$; eritque $= \int \frac{-x^3 dx}{(1-x^2)^{\frac{7}{2}}} = \int \frac{(1-u^2)\, u\, du}{u^7} = \int \frac{du}{u^6} - \int \frac{du}{u^4} = -\frac{1}{5u^5} + \frac{1}{3u^3} = -\frac{1}{5(1-x^2)^{\frac{5}{2}}} + \frac{1}{3(1-x^2)^{\frac{3}{2}}} = -\frac{1}{5\sin\omega^5} + \frac{1}{3\sin\omega^3} + \varphi$. Q. E. I.

Schol. V. $\varphi = \infty - \infty = 0$; ergo $\int \frac{d\omega \cos\omega^3}{\sin\omega^6} = -\frac{1}{5\sin\omega^5} + \frac{1}{3\sin\omega^3}$.

Esto formula sexta $\int \frac{d\omega}{\sin\omega}$, quæ æquatur $\int \frac{-dx}{1-x^2} = L\sqrt{\frac{1-x}{1+x}} = L\frac{1-x}{\sqrt{1-x^2}}$.

Jam ex supra demonstratis $\frac{1-x}{\sqrt{1-x^2}} = tang$

$\frac{1}{2}\omega$; ergo $S\frac{d\omega}{\sin\omega} = L\,\mathrm{tang}\,\frac{1}{2}\omega + \phi$ Q.E.I.

Schol. VI. Facta $\omega = 0$ prodit $\phi = -L0 = -\times-\infty = \infty$; proindeque $S\frac{d\omega}{\sin\omega} = \infty$.

Esto tandem formula postrema $S\frac{d\omega}{\sin\omega^{m}}$, quæ substitutione facta degenerat in $S\frac{-dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$. Ut formula hæc $S\frac{-dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ ad integrationem revocetur, conferatur cum formula Lemmatis III. $S\frac{dz}{\overline{1-z^2}^{p}}$; eritque $p = \frac{m+1}{2}$; & quoniam (Lemm. III.) $S\frac{dz}{\overline{1-z^2}^{p}} = \frac{z}{2p-2\times\overline{1-z^2}^{p-1}} + \frac{2p-3}{2p-2}S\frac{dz}{\overline{1-z}^{2p-1}}$; erit

$$S\frac{-dx}{\overline{1-x^2}^{\frac{m+1}{2}}} = \frac{-x}{m-1\times\overline{1-x^2}^{\frac{m-1}{2}}} + \frac{m-2}{m-1}$$

$$S\frac{-dx}{\overline{1-x^2}^{\frac{m-1}{2}}} = \frac{-\cos\omega}{m-1\times\sin\omega^{m-1}} + \frac{m-2}{m-1}$$

$$S\frac{d\omega}{\sin\omega^{m-2}}.$$ Q. E. I.

Absoluta jam Eulerianarum formularum integratione, ad Formulas quatuor canonicas a Geometrarum nemine hactenus, quod sciam, in-



integratas progredior. Sed antequam harum integrationem aggrediamur, sequentia Theoremata præmittere operæ pretium esse arbitramur.

### THEOREMA I.

$$S\frac{z^n dz}{\overline{1+bz^m}^{p}} = \frac{z^{n+1}}{\overline{p-1}\times m\times\overline{1+bz^m}^{p-1}} + \frac{\overline{p-1}\times m-n-1}{\overline{p-1}\times m}$$

$$S\frac{z^n dz}{\overline{1+bz^m}^{p-1}}$$

### DEM.

Supponatur $S\frac{z^n dz}{\overline{1+bz^m}^{p}} = \frac{M}{\overline{1+bz^m}^{p-1}} + C$ $S\frac{z^h dz}{\overline{1+bz^m}^{p-1}}$, in qua æquatione M designat functionem ipsius z determinandam, C & h constantes inveniendas. Facta differentiatione obtinebitur $\frac{z^n dz}{\overline{1+bz^m}^{p}} =$

$$\frac{dM\times\overline{1+bz^m} - \overline{p-1}\times mbz^{m-1}dzM}{\overline{1+bz^m}^{p}} + \frac{Cz^h dz}{\overline{1+bz^m}^{p-1}},$$

seu $bz^m dM + dM - \overline{p-1}\times mbMz^{m-1}dz + Cbz^{h+m}dz + Cz^h dz - z^n dz = 0$. Jam hæc æquatio subsistit, si M sit simplex potestas ipsius z, idest $M = Az^r$; facta enim subrogatione, æquatio superior in hanc degenerabit $rbAz^{m+r-1} + rAz^{r-1} - \overline{p-1}\times mbAz^{m+r-1} + cbz^{m+h} + cz^h - z^n = 0$. Porro r, & h possunt

ſunt ita determinari, ut tres æquationis termini $rbAz^{m+r-1}$, $-\overline{p-1}\times mbAz^{m+r-1}$, $Cbz^{h+m}$, aliique tres $rAz^{r-1}$, $Cz^{h}$, $-z^{n}$ eandem ipſius z poteſtatem contineant. In hac autem hypotheſi erit $m+r-1=h+m$, ſeu $r-1=h$, & $h=n$, adeoque $r=n+1$. Itaque factis $h=n$, & $r=n+1$ obtinebitur, quod petebatur, & æquatio ad duos terminos redigetur $\overline{n+1}\times bA-\overline{p-1}\times mbA+cb\times z^{m+n}+\overline{n+1}\times A+c-1\times z^{n}=0$; & coefficientibus nihilo æquatis invenientur c, & A, ſcilicet $A=\frac{1}{\overline{p-1}\times m}$, $C=\frac{\overline{p-1}\times m-n-1}{\overline{p-1}\times m}$; hinc $M=Az^{r}=\frac{z^{n+1}}{\overline{p-1}\times m}$, ac denique

$$S\frac{z^{n}dz}{\overline{1+bz^{m}}^{p}}=\frac{M}{\overline{1+bz^{m}}^{p-1}}+C\,S\frac{z^{h}dz}{\overline{1+bz^{m}}^{p-1}}=\frac{z^{n+1}}{\overline{p-1}\times m\times\overline{1+bz^{m}}^{p-1}}+\frac{\overline{p-1}\times m-n-1}{\overline{p-1}\times m}\,S\frac{z^{n}dz}{\overline{1+bz^{m}}^{p-1}}.$$

Q. F. D.

## THEOREMA II.

Eadem formula $S\frac{z^{n}dz}{\overline{1+bz^{m}}^{p}}$ invenitur etiam æqualis

$$-\frac{z^{n-m+1}}{bm\times\overline{p-1}\times\overline{1+bz^{m}}^{p-1}}+\frac{n-m+1}{bm\times\overline{p-1}}\,S\frac{z^{n-m}dz}{\overline{1+bz^{m}}^{p-1}}$$

DEM.



## DEM.

Æquatio $bz^{m}dM+dM-\overline{p-1}\times mbMz^{m-1}dz+cbz^{h+m}dz+cz^{h}dz-z^{n}dz=0$ in præcedenti Theoremate inventa ſubſiſtere poteſt etiam ubi r diverſam quantitatem deſignat, ſcilicet ubi $M=Az^{r}$ diverſam exprimit ipſius z poteſtatem. Itaque in æquatione altera $rbAz^{m+r-1}-\overline{p-1}\times mbAz^{m+r-1}+cbz^{h+m}-z^{n}+rAz^{r-1}+cz^{h}=0$, r, & h ita determinari poſſunt, ut quatuor priores æquationis termini eandem ipſius z potentiam complectantur, & eandem rurſus contineant reliqui duo. Prodibit vero $m+r-1=h+m$, $r-1=h$, & $h+m=n$, ſeu $h=n-m$, & $r=n-m+1$. Inventis porro r, & h æquatio duos tantum terminos complectetur, ideſt $\overline{n-m+1}-\overline{p-1}\times m\times bA+cb-1\times z^{n}+\overline{n-m+1}\times A+c\times z^{n-m}=0$

Ex hac autem æquatione (coefficientibus nihilo æquatis) obtinetur $A=\frac{-1}{bm\times\overline{p-1}}$, & $C=\frac{n-m+1}{bm\times\overline{p-1}}$; unde ſtatim eruitur

$$S\frac{z^{n}dz}{\overline{1+bz^{m}}^{p}}=\frac{M}{\overline{1+bz^{m}}^{p-1}}+C\,S\frac{z^{h}dz}{\overline{1+bz^{m}}^{p-1}}=\frac{-z^{n-m+1}}{bm\times\overline{p-1}\times\overline{1+bz^{m}}^{p-1}}+\frac{n-m+1}{bm\times\overline{p-1}}\times S\frac{z^{n-m}dz}{\overline{1+bz^{m}}^{p-1}}.$$

Q. E. D.

Coroll.

Coroll. Ex hoc Theoremate eruitur illius demonstratio, quod magnus Eulerus indemonstratum relinquit in *Commentariis Petropolitanis* tom. VI. *De constructione Æquationis differentialis* $ax^n dx = dy + yydx$. Theorema Eulerianum est ejusmodi $S\frac{z^{\theta\mu}dz}{\overline{1+bz^\mu}^{\nu+\theta}} =$

$$-\frac{z^{\overline{\theta-1}.\mu+1}}{b\mu.\overline{\nu+\theta-1}.\overline{1+bz^\mu}^{\nu+\theta-1}} + \frac{\overline{\theta-1}.\mu+1}{b\mu.\overline{\nu+\theta-1}}$$

$S\frac{z^{\overline{\theta-1}.\mu}}{\overline{1+bz^\mu}^{\nu+\theta-1}}$. Comparata Euleriana formula $S\frac{z^{\theta\mu}dz}{\overline{1+bz^\mu}^{\nu+\theta}}$ cum nostra $S\frac{z^n dz}{\overline{1+bz^m}^p}$ obtinebitur $n = \theta\mu$, $p = \nu+\theta$, $m = \mu$; proindeque quoniam (Theor. II.) $S\frac{z^n dz}{\overline{1+bz^m}^p} =$

$$-\frac{z^{n-m+1}}{bm\times\overline{p-1}\times\overline{1+bz^m}^{p-1}} + \frac{n-m+1}{bm\times\overline{p-1}} S\frac{z^{n-m}dz}{\overline{1+bz^m}^{p-1}},$$

facta exponentium subrogatione, $S\frac{z^{\theta\mu}dz}{\overline{1+bz^\mu}^{\nu+\theta}}$ prodibit $= -\frac{z^{\overline{\theta-1}.\mu+1}}{b\mu.\overline{\nu+\theta-1}.\overline{1+bz^\mu}^{\nu+\theta-1}}$

$$+\frac{\overline{\theta-1}.\mu+1}{b\mu.\overline{\nu+\theta-1}} S\frac{z^{\overline{\theta-1}.\mu}}{\overline{1+bz^\mu}^{\nu+\theta-1}}$$





THEOREMA III.

$$S\frac{z^h dz}{\overline{1+bz^m}^r} = \frac{z^{h+1}}{\overline{h+1}\times\overline{1+bz^m}^r} + \frac{bmr}{h+1} S\frac{z^{h+m}dz}{\overline{1+bz^m}^{r+1}}$$

DEM.

Ex Theoremate præcedenti est $S\frac{z^n dz}{\overline{1+bz^m}^p} =$

$$-\frac{z^{n-m+1}}{bm\times\overline{p-1}\times\overline{1+bz^m}^{p-1}} + \frac{n-m+1}{bm\times\overline{p-1}} S\frac{z^{n-m}dz}{\overline{1+bz^m}^{p-1}};$$

unde infertur $S\frac{z^{n-m}dz}{\overline{1+bz^m}^{p-1}} = \frac{z^{n-m+1}}{\overline{n-m+1}\times\overline{1+bz^m}^{p-1}}$

$+ \frac{bm\times\overline{p-1}}{n-m+1} S\frac{z^n dz}{\overline{1+bz^m}^p}$, Conferatur modo formula $S\frac{z^{n-m}dz}{\overline{1+bz^m}^{p-1}}$ cum formula

$S\frac{z^h dz}{\overline{1+bz^m}^r}$, eritque $n-m = h$, $p-1 = r$. Facta ergo exponentium Subrogatione patet quod propositum est; erit nimirum $S\frac{z^h dz}{\overline{1+bz^m}^r} =$

$$\frac{z^{h+1}}{\overline{h+1}\times\overline{1+bz^m}^r} + \frac{bmr}{h+1} S\frac{z^{h+m}dz}{\overline{1+bz^m}^{r+1}}$$

Q. E. D.

### THEOREMA IV.

$$S\frac{z^h dz}{\overline{1+bz^m}^{\,r}} = \frac{rm}{rm-h-1} S\frac{z^h dz}{\overline{1+bz^m}^{\,r+1}} - \frac{z^{h+1}}{rm-h-1\times\overline{1+bz^m}^{\,r}}.$$

### DEM.

Ex Theoremate I. est $S\frac{z^n dz}{\overline{1+bz^m}^{\,p}} =$

$$\frac{z^{n+1}}{m\times\overline{p-1}\times\overline{1+bz^m}^{\,p-1}} + \frac{\overline{p-1}\times m-n-1}{\overline{p-1}\times m}$$

$S\frac{z^n dz}{\overline{1+bz^m}^{\,p-1}}$; atque inde infertur

$$S\frac{z^n dz}{\overline{1+bz^m}^{\,p-1}} = \frac{\overline{p-1}\times m}{\overline{p-1}\times m-n-1} S\frac{z^n dz}{\overline{1+bz^m}^{\,p}}$$

$\frac{-z^{n+1}}{\overline{p-1}\times \overline{m-n-1}\times\overline{1+bz^m}^{\,p-1}}$; proindeque factis $n=h$, $p-1=r$, orietur $S\frac{z^h dz}{\overline{1+bz^m}^{\,r}} = \frac{rm}{rm-h-1}\times$

$$S\frac{z^h dz}{\overline{1+bz^m}^{\,r+1}} - \frac{z^{h+1}}{rm-h-1\times\overline{1+bz^m}^{\,r}}\text{ Q.E.D.}$$

Accedamus nunc ad Formularum integratio-



tionem. Sumpto, ut supra, $x$ pro cosinu arcus $\omega$ radio $1$ descripti, prodit

I. $S\frac{d\omega \sin\omega^n}{\cos\omega^m} = S\frac{-dx\times\overline{1-x^2}^{\,\frac{n-1}{2}}}{x^m}$

II. $S\frac{d\omega\cos\omega^n}{\sin\omega^m} = S\frac{-x^n dx}{\overline{1-x^2}^{\,\frac{m+1}{2}}}$

III. $S\,d\omega\sin\omega^n\cos\omega^m = S-x^m dx\times\overline{1-x^2}^{\,\frac{n-1}{2}}$

IV. $S\frac{d\omega}{\sin\omega^n\cos\omega^m} = S\frac{-dx}{x^m\times\overline{1-x^2}^{\,\frac{n+1}{2}}}$

Esto itaque

### PROBLEMA I.

Formulam I. $\frac{d\omega\sin\omega^n}{\cos\omega^m}$ integrare, quando exponentes $n$, $m$ sunt numeri integri, sive positivi, sive negativi, atque etiam nihilum

### SOLUTIO

Est $\frac{d\omega\sin\omega^n}{\cos\omega^m} = -\frac{dx\times\overline{1-x^2}^{\,\frac{n-1}{2}}}{x^m}$. Jam vero $\frac{-dx\times\overline{1-x^2}^{\,\frac{n-1}{2}}}{x^m}$, quando $m$, $n$ sunt numeri integri, sive positivi sint, sive negativi, atque etiam nihilum, facile per notas regulas ad ad integrationem revocatur. Vel enim $n$ par eſt, vel impar. Si $n$ eſt numerus impar, $n-1$ erit par; proindeque $\frac{n-1}{2}$ erit numerus integer; hinc formula $\frac{-dx\,\overline{1-x^2}^{\frac{n-1}{2}}}{x^m}$, erit e veſtigio integrabilis elevando $1-x^2$ ad poteſtatem integram $\frac{n-1}{2}$. Si vero $n$ ſit numerus par, $\frac{n-1}{2}$ erit fractio, & $\overline{1-x^2}^{\frac{n-1}{2}}$ erit quantitas radicalis, quæ tamen protinus eliminabitur facta de more $\sqrt{1-x^2} = 1 - zx$; quo facto obtinebitur integrale per regulas notas; idque facilius eruetur ubi exponentium alteruter $n$, vel $m$ ſit nihilo æqualis; quod ſi uterque ſit $= 0$, formula abibit in hanc $\frac{-dx}{\sqrt{1-x^2}}$, cujus integrale eſt arcus $\omega$, ut patet. At quoniam communis hæc integrandi methodus, ubi $\frac{n-1}{2}$ eſt fractio, laboris eſt, ac moleſtiæ pleniſsima, præſtabit methodum ſequentem adhibere.

Eſto itaque eadem formula $\frac{-dx\,\overline{1-x^2}^{\frac{n-1}{2}}}{x^m}$, cujus loco ſcribi poteſt $\frac{-x^{-m}dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$. In hac autem $m$ vel minor erit binario, vel major, (ſupponuntur interim $m$, & $n$ poſitivi). Si $m$ eſt binario minor, vel erit 1, vel nihilum. Ponatur ergo primo $m=1$, &

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formula mutabitur in hanc $\frac{-x^{-1}dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$, quæ per ſubſtitutionem ſimpliciſſimam $\sqrt{1-x^2} = u$, abit in $\frac{u^n du}{1-u^2}$, quæ nullum negotium faceſſit. Ponatur ſecundo $m=0$, eritque formula $\frac{-dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$; quæ conferatur cum formula Theorematis IV.

$\frac{z^h dz}{\overline{1+bz^m}^r}$, Eritque $h=0$, $r=\frac{1-n}{2}$, $m=2$, $b=-1$. Eſt autem (Theor. IV.)

$$S\frac{z^h dz}{\overline{1+bz^m}^r} = \frac{rm}{rm-h-1}S\frac{z^h dz}{\overline{1+bz^m}^{r+1}} - \frac{z^{h+1}}{rm-h-1\times\overline{1+bz^m}^r}.$$

Ergo facta ſubſtitutione erit

$$S\frac{-dx}{\overline{1-x^2}^{\frac{1-n}{2}}} = \frac{n-1}{-n}S\frac{dx}{\overline{1-x^2}^{\frac{3-n}{2}}}$$

 —x

$$-\frac{x}{n\times\overline{1-x^2}^{\frac{1-n}{2}}} = \frac{n-1}{n} S\frac{-dx}{\overline{1-x^2}^{\frac{3-n}{2}}}$$

$$-\frac{x}{n\times\overline{1-x^2}^{\frac{1-n}{2}}} = \frac{n-1}{n} S d\omega \sin\omega^{n-2}$$

$-\frac{\cos\omega \sin\omega^{n-1}}{n}$. Rursus ex eodem IV. Theoremate erit $\frac{n-1}{n} S d\omega \sin\omega^{n-2} =$

$$\frac{n-1}{n} S\frac{-dx}{\overline{1-x^2}^{\frac{3-n}{2}}} = \frac{\overline{n-1}\times\overline{n-3}}{n\times\overline{n-2}}$$

$$S\frac{-dx}{\overline{1-x^2}^{\frac{5-n}{2}}} - \frac{\overline{n-1}\times x}{n\times\overline{n-2}\times\overline{1-x^2}^{\frac{3-n}{2}}} =$$

$$\frac{\overline{n-1}\times\overline{n-3}}{n\times\overline{n-2}} S d\omega \sin\omega^{n-4} -$$

$\frac{\overline{n-1}\times\cos\omega.\sin\omega^{n-3}}{n\times\overline{n-2}}$; Ergo $S\frac{-dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$

$$S d\omega \sin\omega^n = \frac{\overline{n-1}\times\overline{n-3}}{n\times\overline{n-2}} \times$$

S d



$$S d\omega \sin\omega^{n-4} - \frac{\cos\omega \sin\omega^{n-1}}{n}$$

$-\frac{\overline{n-1}\times\cos\omega \sin\omega^{n-3}}{n\times\overline{n-2}}$. Eadem ratione, quoniam ex hypothesi n est numerus par, usurpando semper idem IV. Theorema, obtinebitur integrale formulæ $\frac{-dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$ datum per sinum, & cosinum arcus $\omega$, & per arcum ipsum $\omega$. Sit ex: gr. $n = 2$, eritque

$$S\frac{-dx}{\overline{1-x^2}^{\frac{1-n}{2}}} = \frac{n-1}{n} S d\omega \sin\omega^{n-2} -$$

$\frac{\cos\omega \sin\omega^{n-1}}{n} = \frac{1}{2}\omega - \frac{1}{2}\cos\omega \sin\omega$. Sit

$n=4$, eritque $S\frac{-dx}{\overline{1-x^2}^{\frac{1-n}{2}}} = \frac{\overline{n-1}\times\overline{n-3}}{n\times\overline{n-2}}$

$$S d\omega \sin\omega^{n-4} - \frac{\cos\omega \sin\omega^{n-1}}{n} - \frac{\overline{n-1}\times\cos\omega \sin\omega^{n-3}}{n\times\overline{n-2}}$$

$= \frac{3}{8}\omega - \frac{1}{4}\cos\omega \sin\omega^3 - \frac{3}{8}\cos\omega \sin\omega$; atque ita porro.

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Ponatur modo $m$ binario major, qui vel par erit, vel impar; in utroque autem casu ope Theorematis II. eruetur integrale. Conferatur ergo formula Theorematis III.

$$\int \frac{z^{h}\,dz}{\left(1+bz^{m'}\right)^{r}}$$ cum formula $$\int \frac{-x^{-m}\,dx}{\left(1-x^{2}\right)^{\frac{1-n}{2}}};$$

Eritque $h = m$, $r = \frac{1-n}{2}$, $m' = 2$, $b = -1$. Hinc quoniam (Theor. III.)

$$\int \frac{z^{h}\,dz}{\left(1+bz^{m'}\right)^{r}} = \frac{bm'r}{h+1}\int \frac{z^{h+m'}\,dz}{\left(1+bz^{m'}\right)^{r+1}} + \frac{z^{h+1}}{(h+1)\times\left(1+bz^{m'}\right)^{r}},$$ facta subrogatione erit

$$\int \frac{-x^{-m}\,dx}{\left(1-x^{2}\right)^{\frac{1-n}{2}}} = \frac{n-1}{1-m}\int \frac{-x^{2-m}\,dx}{\left(1-x^{2}\right)^{\frac{3-n}{2}}}$$

$$+ \frac{x^{1-m}}{(m-1)\times\left(1-x^{2}\right)^{\frac{1-n}{2}}} = \frac{n-1}{1-m}\int \frac{d\omega \sin\omega^{n-2}}{\cos\omega^{m-2}} +$$

$$\frac{\sin\omega^{n-1}}{(m-1)\times\cos\omega^{m-1}}.$$ Rursus ex eodem Theore-

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mate prodibit $$\frac{n-1}{1-m}\int \frac{-x^{2-m}\,dx}{\left(1-x^{2}\right)^{\frac{3-n}{2}}} =$$

$$\frac{(n-1)\times(n-3)}{(1-m)\times(3-m)}\int \frac{-x^{4-m}\,dx}{\left(1-x^{2}\right)^{\frac{5-n}{2}}} -$$

$$\frac{(n-1)\times x^{3-m}}{(m-1)\times(m-3)\times\left(1-x^{2}\right)^{\frac{3-n}{2}}}.$$ Itaque

$$\int \frac{-x^{-m}\,dx}{\left(1-x^{2}\right)^{\frac{1-n}{2}}} = \frac{(n-1)\times(n-3)}{(m-1)\times(m-3)}\int \frac{-x^{4-m}\,dx}{\left(1-x^{2}\right)^{\frac{5-n}{2}}} +$$

$$\frac{x^{1-m}}{(m-1)\times\left(1-x^{2}\right)^{\frac{1-n}{2}}} - \frac{(n-1)\times x^{3-m}}{(m-1)\times(m-3)\times\left(1-x^{2}\right)^{\frac{3-n}{2}}} =$$

$$\frac{(n-1)\times(n-3)}{(m-1)\times(m-3)} \times \int \frac{d\omega \sin\omega^{n-4}}{\cos\omega^{m-4}} + \frac{\sin\omega^{n-1}}{(m-1)\times\cos\omega^{m-1}}$$

$$\frac{-(n-1)\times\sin\omega^{n-3}}{(m-1)\times(m-3)\times\cos\omega^{m-3}}.$$ Eadem ratione usurpando semper Theorema III. numero vicium $= \frac{m}{2}$, si $m$ est par, & vicibus $\frac{m-1}{2}$, si $m$ est impar, obtinebitur integrale for-

 mulæ

mulæ $\frac{x^{-m}dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$ datum per ſinum, & coſinum arcus $\omega$, & ſupererit

$$S\frac{-dx}{\overline{1-x^2}^{\frac{m+1-n}{2}}}$$, quando m eſt numerus par, vel $S\frac{-x^{-1}dx}{\overline{1-x^2}^{\frac{m-n}{2}}}$, quando m eſt impar. Jamvero, ſi $m = n$,

$$S\frac{-dx}{\overline{1-x^2}^{\frac{m+1-n}{2}}} = \omega$$; ſi $m > n$, ſeu $m + 1 - n$ eſt numerus poſitivus, tunc

$$S\frac{-dx}{\overline{1-x^2}^{\frac{m+1-n}{2}}}$$ dabitur per ſinum, & coſinum arcus $\omega$, & per arcum ipſum $\omega$, modo adhibeatur Theorema I. Si vero $m < n$, ſive $m + 1 - n$ ſit numerus negativus (hic enim ſemper ſupponitur n par), tunc

tunc



tunc adhibito Theoremate IV. dabitur rurſus $S\frac{-dx}{\overline{1-x^2}^{\frac{m+1-n}{2}}}$ per ſinum, & coſinum arcus $\omega$, & per arcum eundem $\omega$. Quod vero ſpectat ad Formulam

$$S\frac{-x^{-1}dx}{\overline{1-x^2}^{\frac{m-n}{2}}}$$, a qua pendet propoſitæ Formulæ integratio, quando m eſt numerus impar; hæc per ſimpliciſſimam ſubſtitutionem $\sqrt{1-x^2} = u$ degenerat in

$$\frac{u^{-m+1+n}du}{1-u^2}$$, quæ, ut conſtat, facillime integratur, & a logarithmis, ſive ab hyperbolæ quadratura dependet. En igitur Formulam I. $\frac{d\omega \sin\omega^n}{\cos\omega^m}$ integratam in caſibus omnibus, in quibus m, & n ſint numeri integri poſitivi. Conſulto omittimus caſum alium, quo exponentium alteruter, vel uterque negativus aſſumitur, quia tunc For-



mula

mula $\frac{d\omega \sin\omega^n}{\cos\omega^m}$ mutatur in alias tres inferius integrandas. Si enim negativus sit m, Formula superior mutatur in $d\omega \sin\omega^n \cos\omega^m$; quæ congruit cum Formula III.; si negativus sit n, mutatur in $\frac{d\omega}{\sin\omega^n \cos\omega^m}$, quæ coincidit cum IV.; si ambo sint negativi, abit in $\frac{d\omega \cos\omega^m}{\sin\omega^n}$, quæ revocatur ad II. Ergo integrale Formulæ I. $\frac{d\omega \sin\omega^n}{\cos\omega^m}$ semper invenietur, dabiturque per solum cosinum arcus ω, quando n est numerus impar (quicumque sit m); dabitur per sinum arcus ω, & per sinus ipsius logarithmos, quando n est par, m=1; dabitur per sinum, & cosinum arcus ω, & per arcum ipsum ω, quando n est par, m=0; dabitur per sinum, & cosinum arcus ω, & per arcum eundem ω, quando n est par, m=2; dabitur pariter per sinum, & cosinum arcus ω, & per arcum ipsum ω, quando n par = m; dabitur rursus per sinum, & cosinum arcus ω, & per arcum ω, quando m par > n pari, vel m par < n pari; dabitur per sinum, & cosinum arcus ω, & per sinus ipsius logarithmos, quando n est par, & m impar > 1; dabitur tandem

per



per solum arcum ω, quando m, & n sunt nihilo æquales. Q. E. F.

Progredior nunc ad integrationem Formulæ IV. $\frac{d\omega}{\sin\omega^m \cos\omega^n}$, quia ex hac partim pendet ceterarum integratio. Esto itaque

PROBLEMA II.

Formulam IV. $\frac{d\omega}{\sin\omega^m \cos\omega^n}$ integrare.

SOLUTIO

Est $\frac{d\omega}{\sin\omega^m \cos\omega^n} = \frac{-dx}{x^n \times \overline{1-x^2}^{\frac{m+1}{2}}}$. Jam $\frac{-dx}{x^n \times \overline{1-x^2}^{\frac{m+1}{2}}}$ protinus integratur per notas regulas, si m sit numerus impar, adeoque $\frac{m+1}{2}$ numerus integer, vel etiam per Theorema I.: collata enim formula

$$\frac{-dx}{x^n \times \overline{1-x^2}^{\frac{m+1}{2}}},$$ vel

—x

$\frac{-x^{-n}dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ cum formula $\frac{z^{n'}dz}{\overline{1+bz^{m'}}^{p}}$

I. Theorematis, factisque $n' = -n$, $m' = 2$, $p = \frac{m+1}{2}$, $b = -1$; prodibit facta comparatione

$$S\frac{-x^{-n}dx}{\overline{1-x^2}^{\frac{m+1}{2}}} = \frac{-x^{-n+1}}{m-1 \times \overline{1-x^2}^{\frac{m-1}{2}}} +$$

$$\frac{m+n-2}{m-1} S\frac{-x^{-n}dx}{\overline{1-x^2}^{\frac{m-1}{2}}} =$$

$$\frac{-1}{m-1 \times \cos\omega^{n-1} \sin\omega^{m-1}} + \frac{m+n-2}{m-1} S\frac{d\omega}{\sin\omega^{m-2} \cos\omega^{n}};$$

atque ita procedendo invenietur

$S\frac{-x^{-n}dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ data per sinum, & cosinum arcus $\omega$ & per $S\frac{-x^{-n}dx}{1-x^2}$, quæ per notas regulas statim integratur. Si vero m sit numerus par, $\frac{m+1}{2}$ erit fractio, continebitque Formula quantitatem radicalem. Tunc vero vel erit

n bi-



n Binario minor, vel æqualis, vel major. Si n est binario minor, vel erit $= 0$, vel $= 1$; si $n = 0$, Formula mutabitur in

$S\frac{-dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$, quæ per Theorema I.

dabitur per sinum, & cosinum arcus $\omega$, & per arcum ipsum $\omega$; si $n = 1$, formula

abibit in $S\frac{x^{-1}dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$, quæ per substitutionem $\sqrt{1-x^2} = u$, fit $= S\frac{du}{u^m \times 1-u^2}$

per notas regulas facile integrabili. Si $n = 2$, vel $n > 2$, comparata Formula

$S\frac{-x^{-n}dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ cum formula $S\frac{z^h dz}{\overline{1+bz^{m'}}^{r}}$

Theorematis III. erit $h = -n$, $r = \frac{m+1}{2}$, $m' = 2$, $b = -1$; hinc quoniam (Theor. III.)

S z h

$$S\frac{z^{h}\,dz}{(1+bz^{m})^{r}} = \frac{z^{h+1}}{(h+1)\times(1+bz^{m})^{r}} + \frac{bmr}{h+1}\times S\frac{z^{h+m}\,dz}{(1+bz^{m})^{r+1}}$$, facta substitutione erit

$$S\frac{-x^{-n}\,dx}{(1-x^{2})^{\frac{m+1}{2}}} = \frac{m+1}{n-1}\times S\frac{-x^{-n+2}\,dx}{(1-x^{2})^{\frac{m+3}{2}}} + \frac{x^{-n+1}}{(n-1)\times(1-x^{2})^{\frac{m+1}{2}}},$$

proindeque, ubi $n=2$, erit $S\frac{-x^{-n}\,dx}{(1-x^{2})^{\frac{m+1}{2}}}$

$$= \frac{1}{\cos\omega^{n-1}\sin\omega^{m+1}} + (m+1)\,S\frac{-dx}{(1-x^{2})^{\frac{m+3}{2}}}.$$

Porro $(m+1)\,S\frac{-dx}{(1-x^{2})^{\frac{m+3}{2}}}$ per Theorema I. invenietur data per sinum, & cosinum arcus ω, & per eundem arcum

cum



cum ω. Instituta enim hujus Formulæ comparatione cum formula I. Theorematis eruetur $(m+1)\,S\frac{-dx}{(1-x^{2})^{\frac{m+3}{2}}} = \frac{-x}{(1-x^{2})^{\frac{m+1}{2}}}$

$+ m\,S\frac{-dx}{(1-x^{2})^{\frac{m+1}{2}}} = \frac{-\cos\omega}{\sin\omega^{m+1}} +$

$m\,S\frac{-dx}{(1-x^{2})^{\frac{m+1}{2}}}$; atque ita porro adhibito Theoremate I. numero vicium $= \frac{m}{2}$, obtinebitur $S\frac{-x^{-2}\,dx}{(1-x^{2})^{\frac{m+1}{2}}}$ data per sinum, & cosinum arcus ω, & per $S\frac{-dx}{\sqrt{1-x^{2}}}$, seu per arcum ω. Si autem n sit binario major, & par, tunc adhibito Theoremate III. vicibus $\frac{n}{2}$ invenietur $S\frac{-x^{-n}\,dx}{(1-x^{2})^{\frac{m+1}{2}}}$ data per sinum, & cosinum arcus ω, & per

S—dx

$S\frac{-dx}{\overline{1-x^2}^{\frac{m+n+1}{2}}}$, quæ per Theorema I. vicibus $= \frac{m+n}{2}$ usurpatum dabitur per sinum, & cosinum arcus $\omega$, & per arcum ipsum $\omega$. Si n binario major sit numerus impar, tunc usurpato vicibus $\frac{n-1}{2}$ Theoremate III. dabitur $S\frac{-x^{-n}dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ per sinum, & cosinum arcus $\omega$, & per $S\frac{-x^{-1}dx}{\overline{1-x^2}^{\frac{m+n}{2}}}$, quæ per consuetam substitutionem $\sqrt{1-x^2} = u$, mutabitur in $S\frac{du}{\overline{1-u^2}\times u^{m+n-1}}$ nulla molestia integrabilem. Omitto casus, in quibus exponentium m, & n alteruter, vel uterque negativus assumitur, quia tum Formulæ aliæ recurrunt. Q. E. F.

PRO-



## PROBLEMA III.

Formulam II. $\frac{d\omega \cos\omega^n}{\sin\omega^m}$, vel $\frac{-x^n dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ integrare.

### SOLUTIO

Si m sit numerus impar, quoniam tunc $\frac{m+1}{2}$ est numerus integer, Formula $\frac{-x^n dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ integrabitur vel per notas regulas, vel promptius per Theorema II. Si vero m sit numerus par, tum n vel erit $=0$, vel $=1$, vel $=2$, vel $>2$; si $n=0$, Formula abit in $\frac{-dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$, quam in præcedenti Problemate per Theorema I. integravimus; si $n=1$, Formula erit $\frac{-x\,dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$, quæ summam habet algebraicam, scilicet $\frac{-1}{\overline{m-1}\times\overline{1-x^2}^{\frac{m-1}{2}}} =$

$-1$

$\frac{-1}{m-1 \times \sin\omega^{m-1}}$. Si $n=2$, ope II. Theorematis invenietur $S\frac{-x^n\,dx}{\overline{1-x^2}^{\frac{m+1}{2}}} =$

$$\frac{-x}{m-1 \times \overline{1-x^2}^{\frac{m-1}{2}}} - \frac{1}{m-1} \times S\frac{dx}{\overline{1-x^2}^{\frac{m-1}{2}}}.$$

Porro $\frac{-1}{m-1} S\frac{dx}{\overline{1-x^2}^{\frac{m-1}{2}}}$ per Theorema I. vicibus $\frac{m-2}{2}$ usurpatum invenietur de more pendens ab $S\frac{dx}{\sqrt{1-x^2}}$, nimirum ab arcu $\omega$. Denique si $n > 2$ sit numerus par, tunc usurpato vicibus $\frac{n}{2}$ Theoremate II. invenietur Formula $S\frac{x^n\,dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ data per sinum, & cosinum arcus $\omega$, & per $S\frac{dx}{\overline{1-x^2}^{\frac{m-n+1}{2}}}$, quæ datur rursus per





sinum, & cosinum arcus $\omega$, & per eundem arcum $\omega$; idque obtinebitur per Theorema I., quando $m > n$, seu $m - n + 1$ est numerus positivus; & per Theorema IV., ubi $m$ sit $< n$, seu $m - n + 1$ sit numerus negativus. At si $n$ binario major sit impar, usurpato vicibus $\frac{n-1}{2}$ Theoremate II. dabitur $S\frac{x^n\,dx}{\overline{1-x^2}^{\frac{m+1}{2}}}$ per sinum, & cosinum arcus $\omega$, ac præterea per $S\frac{x\,dx}{\overline{1-x^2}^{\frac{m-n+2}{2}}}$ quæ algebraicam recipit summam, idest

$$\frac{\overline{1-x^2}^{\frac{n-m}{2}}}{n-m} = \frac{\sin\omega^{n-m}}{n-m}.$$

Consulto prætermitto casus alios, in quibus $m$, aut $n$, vel uterque negativus est, quia tum Formula hæc in tres alias degenerat. Q. E. F.

### PROBLEMA IV.

Formulam III. $d\omega \sin\omega^n \cos\omega^m$, seu $-x^m\,dx \times \overline{1-x^2}^{\frac{n-1}{2}}$ integrare.

## SOLUTIO

Sit n numerus impar, erit $\frac{n-1}{2}$ numerus integer, adeoque elevata quantitate $1-x^2$ ad potestatem integram $\frac{n-1}{2}$ habebitur protinus integrale formulæ $-x^m dx \times \overline{1-x^2}^{\frac{n-1}{2}}$, ut per se patet. Si n est numerus par, tum vel m erit $=1$, vel $=0$, vel $=2$, vel $>2$; si $m=1$, prodibit $S - x^m dx \times \overline{1-x^2}^{\frac{n-1}{2}} =$

$$\frac{\overline{1-x^2}^{\frac{n+1}{2}}}{n+1} = \frac{\text{fin}\,\omega^{n+1}}{n+1}$$

. Si vero $m=0$, formula $-x^m dx \times \overline{1-x^2}^{\frac{n-1}{2}} = \frac{-x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$ ope IV. Theorematis facile integratur, dabiturque ejus integrale per finum, & cofinum arcus $\omega$, & per arcum ipfum $\omega$. Si $m=2$, tunc ufurpabitur Theorema II, eritque

$$S\frac{-x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}} = \frac{-x}{n+1 \times \overline{1-x^2}^{\frac{-n-1}{2}}} +$$

$\frac{1}{n+1}$



$$\frac{1}{n+1} S\frac{-dx}{\overline{1-x^2}^{\frac{-n-1}{2}}} = \frac{\cos\omega\,\text{fin}\,\omega^{n+1}}{n+1} +$$

$$\frac{1}{n+1} S\frac{-dx}{\overline{1-x^2}^{\frac{-n-1}{2}}}$$

. Porro $\frac{1}{n+1}$

$$S\frac{-dx}{\overline{1-x^2}^{\frac{-n-1}{2}}}$$

ope IV. Theorematis vicibus $\frac{n+2}{2}$ ufurpati data invenietur per finum, & cofinum arcus $\omega$, & per arcum ipfum $\omega$. Denique fi $m>2$ fit numerus par, ufurpato vicibus $\frac{m}{2}$ Theoremate II. invenietur Formula $S\frac{-x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$ data per finum, & cofinum arcus $\omega$, ac præterea per $S\frac{-dx}{\overline{1-x^2}^{\frac{1-m-n}{2}}}$, quæ per Theorema IV. vicibus $\frac{n+m}{2}$ ufurpatum prodibit rurfus expreffa per finum, & cofinum arcus $\omega$, & per arcum eundem $\omega$, ut tentanti apparebit. Si vero m binario major fit numerus impar, ufurpato Theoremate II. vicibus $\frac{m-1}{2}$

prodibit $S\frac{x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$ expressa per sinum, & cosinum arcus $\omega$, atque insuper per $S\frac{x dx}{\overline{1-x^2}^{\frac{2-m-n}{2}}}$ $S - x dx \times \overline{1-x^2}^{\frac{m+n-2}{2}} =$

$$\frac{-\overline{1-x^2}^{\frac{m+n}{2}}}{m+n} = \frac{\sin\omega^{m+n}}{m+n}$$

; proindeque in hoc casu Formula $\frac{x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}} = d\omega \sin\omega^n$ $\cos\omega^m$ summam recipit algebraicam. Omitto casum, quo exponentium m, n alteruter, vel uterque negativus est, ob rationem alias indicatam. Q. E. F.

### SCHOLIUM GENERALE.

Rite jam integratis propositis Formulis operæ pretium nunc est casus illos investigare, quibus Formulæ eædem algebraicam summam admittunt, & rursus casus alios assignare, quibus earundem integratio vel per circuli rectificationem, vel per hyperbolæ quadraturam obtinetur; ut quæ hactenus detecta sunt uno omnia intuitu facilius comprehendantur, & multiplicato veluti lumine



mine effulgeant. Esto itaque Formula II.

$$S\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}}$$

. Adhibeatur Theorema II., & invenietur Formula hæc positive sumpta (positivam sumo commodi ergo), scilicet

$$S\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}} = \frac{x^{m-1}}{\overline{n-1}\times\overline{1-x^2}^{\frac{n-1}{2}}} - \frac{\overline{m-1}}{n-1}$$

$$S\frac{x^{m-2} dx}{\overline{1-x^2}^{\frac{n-1}{2}}}, \text{ \& } -\frac{\overline{m-1}}{n-1}\times$$

$$S\frac{x^{m-2} dx}{\overline{1-x^2}^{\frac{n-1}{2}}} = -\frac{\overline{m-1}\times x^{m-3}}{\overline{n-1}\times\overline{n-3}\times\overline{1-x^2}^{\frac{n-3}{2}}} +$$

$$\frac{\overline{m-1}\times\overline{m-3}}{\overline{n-1}\times\overline{n-3}} S\frac{x^{m-4} dx}{\overline{1-x^2}^{\frac{n-3}{2}}}$$

; proindeque

$$S\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}} = \frac{x^{m-1}}{\overline{n-1}\times\overline{1-x^2}^{\frac{n-1}{2}}}$$

$$-\frac{\overline{m-1}\times x^{m-3}}{\overline{n-1}\times\overline{n-3}\times\overline{1-x^2}^{\frac{n-3}{2}}} + \frac{\overline{m-1}\times\overline{m-3}}{\overline{n-1}\times\overline{n-3}}$$

$S\dfrac{x^{m-4}dx}{\overline{1-x^2}^{\frac{n-3}{2}}}$ ; atque ita porro procedendo , uſurpato vicibus $= h$ Theoremate II. prodibit $S\dfrac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}} = \dfrac{x^{m-1}}{\overline{n-1} \times \overline{1-x^2}^{\frac{n-1}{2}}}$

$$- \dfrac{\overline{m-1} \times x^{m-3}}{\overline{n-1} \times \overline{n-3} \times \overline{1-x^2}^{\frac{n-3}{2}}} +$$

$$\dfrac{\overline{m-1} \times \overline{m-3} \times x^{m-5}}{\overline{n-1} \times \overline{n-3} \times \overline{n-5} \times \overline{1-x^2}^{\frac{n-5}{2}}} \;\&c. \ldots\ldots\ldots$$

$$\dfrac{\overline{m-1} \times \overline{m-3} \ldots\ldots \times \overline{m-2h+3} \times x^{m-2h+1}}{\overline{n-1} \times \overline{n-3} \times \overline{n-5} \ldots\ldots \times \overline{n-2h+1} \times \overline{1-x^2}^{\frac{n-2h+1}{2}}}$$

$$\pm \dfrac{\overline{m-1} \times \overline{m-3} \ldots\ldots \times \overline{m-2h+1}}{\overline{n-1} \times \overline{n-3} \ldots\ldots \times \overline{n-2h+1}}$$

$S\dfrac{x^{m-2h}dx}{\overline{1-x^2}^{\frac{n-2h+1}{2}}}$. Ex hujus æquationis conſpectu ſtatim innoteſcit Formulam





$S\dfrac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}}$ ſummam recipere algebraicam, ubi m ſit numerus impar , n par , vel n etiam impar , ſed major , quam m, ut videre eſt ſubſtituendo $\frac{m+1}{2}$ loco h , quo facto ultimus æquationis terminus evaneſcit . Si m , & n ſint numeri impares æquales, tum pro h ſubſtituendo $\frac{m-1}{2}$, vel $\frac{n-1}{2}$ prodibit eadem ſumma partim algebraica, partim pendens a $S\dfrac{x\,dx}{\overline{1-x^2}}$, vel a $L\sqrt{\overline{1-x^2}}$.

Si m , & n ſint rurſus impares , ſed m major, pro h ſubrogetur $\frac{n-1}{2}$, pendebitque eadem ſumma a $S\dfrac{x^{m-n+1}dx}{\overline{1-x^2}}$ ; & quia in hac hypotheſi $m-n$ eſt numerus par poſitivus , inſtituta diviſione poteſtatis $x^{m-n+1}$ per $\overline{1-x^2}$ pendebit & ipſa $S\dfrac{x^{m-n+1}dx}{\overline{1-x^2}}$, adeoque & prior

$S\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}}$ ab $S\frac{-x\,dx}{1-x^2}$, nimirum a $L\sqrt{1-x^2}$. Si m sit numerus par, n impar, substituendo itidem $\frac{n-1}{2}$ pro h pendebit iterum

$S\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}}$ ab $S\frac{x^{m-n+1}dx}{\overline{1-x^2}^2}$, quae posterior pendet a L tang $\frac{1}{2}\omega$ si $m=n-1$, vel $m>n-1$; & pendet rursus a logarithmis si $m<n-1$. Superest modo casus postremus, quo ambo m, n sint numeri pares; tum vero substituendo in aequatione superiori $\frac{m}{2}$ pro h invenitur summa superius eruta pendens a

$S\frac{dx}{\overline{1-x^2}^{\frac{n-m+1}{2}}}$, quae pendebit iterum ab arcu $\omega$, si $n=m$, vel $n>m$; si vero $n<m$, tunc adhibito Theoremate IV. eruetur $S\frac{dx}{\overline{1-x^2}^{\frac{n-m+1}{2}}} = \frac{x}{\overline{m-n}\times\overline{1-x^2}^{\frac{n-m+1}{2}}}$ $\frac{\overline{m-n-1}}{\overline{m-n}} S\frac{dx}{\overline{1-x^2}^{\frac{n-m+3}{2}}}$, & usurpato eodem Theoremate numero vicium $=h$, prodibit



$$S\frac{dx}{\overline{1-x^2}^{\frac{n-m+1}{2}}} = \frac{x}{\overline{m-n}\times\overline{1-x^2}^{\frac{n-m+1}{2}}} + \frac{\overline{m-n-1}\times x}{\overline{m-n}\times\overline{m-n-2}\times\overline{1-x^2}^{\frac{n-m+3}{2}}} + \frac{\overline{m-n-1}\times\overline{m-n-3}\times x}{\overline{m-n}\times\overline{m-n-2}\times\overline{m-n-4}\times\overline{1-x^2}^{\frac{n-m+5}{2}}} + \&c.\ldots + \frac{\overline{m-n-1}\times\overline{m-n-3}\ldots\times\overline{m-n-2h+3}\times x}{\overline{m-n}\times\overline{m-n-2}\times\overline{m-n-4}\ldots\times\overline{m-n-2h+2}\times\overline{1-x^2}^{\frac{n-m+2h-1}{2}}} + \frac{\overline{m-n-1}\times\overline{m-n-3}\ldots\times\overline{m-n-2h+1}}{\overline{m-n}\times\overline{m-n-2}\ldots\times\overline{m-n-2h+2}} S\frac{dx}{\overline{1-x^2}^{\frac{n-m+2h+1}{2}}}$$

; in hac autem aequatione subrogando $\frac{m-n}{2}$ pro h, qui in hac hypothesi erit numerus positivus integer, apparebit $S\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}}$ pendere ab

S—

$S\frac{-dx}{\overline{1-x^2}^{\frac{1}{2}}}$, nimirum ab arcu $\omega$.

Haud absimili methodo idem instituetur examen in Formulis reliquis $S\frac{-dx\times\overline{1-x^2}^{\frac{n-1}{2}}}{x^m}$,

$S - x^m dx \times \overline{1-x^2}^{\frac{n-1}{2}}$, $S\frac{-dx}{x^m\times\overline{1-x^2}^{\frac{n+1}{2}}}$;

neque operæ pretium videtur in re per se clara, & evidenti immorari. Ad alia itaque utiliora properamus.

Æquum nunc est, ut Formulæ quatuor propositæ in omnibus casibus integratæ habeantur, casum illum considerare, quo exponentium m, n alteruter, vel uterque fractus assumitur. Esto itaque

## PROBLEMA V.

Formulam I. $\frac{-dx\times\overline{1-x^2}^{\frac{n-1}{2}}}{x^m}$ in hypothesi exponentium fractorum integrare:





## SOLUTIO

Si m est fractio quæcumque, modo n sit numerus impar, proposita Formula protinus per notas regulas integratur; nam elevata quantitate $1-x^2$ ad potestatem integram $\frac{n-1}{2}$, prodit illico integrale, ut per se patet. Integratur pariter proposita Formula, etiam ubi n est fractio quæcumque, modo m sit numerus integer impar $= 2f+1$; etenim posito $n = \frac{h}{r}$, factaque subrogatione $\frac{-}{\overline{1-x^2}^{\frac{1}{2r}}} = y$, orietur

$$\frac{-dx\times\overline{1-x^2}^{\frac{n-1}{2}}}{x^m} = \frac{ry^{r+h-1}dy}{\overline{1-y^{2r}}^{f+1}}$$, quæ per

regulas consuetas haud moleste integratur, ut omnes norunt. Sint modo m, & n simul fractiones. Adhibeatur Theorema I. & invenietur $S\frac{x^{-m}dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$ (assumo Formulam positivam commodi gratia) =

—.

$$-\frac{\overline{1-x^2}^{\frac{n+1}{2}}}{\overline{n+1}\times x^{m-1}} + \frac{\overline{m-n-2}\times\overline{1-x^2}^{\frac{n+3}{2}}}{\overline{n+1}\times\overline{n+3}\times x^{m-1}}$$

$$-\frac{\overline{m-n-2}\times\overline{m-n-4}\times\overline{1-x^2}^{\frac{n+5}{2}}}{\overline{n+1}\times\overline{n+3}\times\overline{n+5}\times x^{m-1}} + \&c.\ldots\ldots\ldots$$

$$\frac{\overline{m-n-2}\times\overline{m-n-4}\times\overline{m-n-6}\ldots\times\overline{m-n-2h+2}\times x^{-m+1}}{\overline{n+1}\times\overline{n+3}\times\overline{n+5}\times\overline{n-7}\ldots\times\overline{n+2h+1}\times\overline{1-x^2}^{\frac{-n-2h+1}{2}}}$$

$$+\frac{\overline{m-n-2}\times\overline{m-n-4}\times\overline{m-n-6}\times\overline{m-n-8}\ldots\times\overline{m-n-2h}}{\overline{n+1}\times\overline{n+3}\times\overline{n+5}\times\overline{n+7}\times\overline{n+9}\ldots\times\overline{n+2h+1}}$$

$S\frac{x^{-m}dx}{\overline{1-x^2}^{\frac{1-n-2h}{2}}}$ usurpato eodem Theoremate vicibus $h$. Ex hujus æquationis conspectu innotescit summam inventam esse algebraicam, si $m-n=2h$, sive fracti sint, sive integri exponentes $m$, $n$, quia tunc evanescit, ut patet, ultimus æquationis terminus. Non moror diutius in hoc Problemate enucleando, quod ea, quæ in sequentibus observabuntur, huic quoque lucem afferent, & splendorem. Esto itaque

PRO-



## PROBLEMA VI.

Formulam II. $\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}}$ in hypothesi eadem exponentium $m$, $n$ fractorum integrare.

### SOLUTIO

Si $m$ est fractio quælibet, $n$ numerus impar, fiat $m=\frac{h}{r}$, $x^{\frac{1}{r}}=y$, eritque $\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}} = \frac{r y^{r+h-1} dy}{\overline{1-y^{2r}}^{\frac{n+1}{2}}}$, quæ per notas regulas haud difficile integratur, ut constat. Pro aliis vero casibus adhibeatur I. Theorema, eritque $S\frac{x^m dx}{\overline{1-x^2}^{\frac{n+1}{2}}} =$

$$\frac{x^{m+1}}{\overline{n-1}\times\overline{1-x^2}^{\frac{n-1}{2}}} + \frac{n-m-2}{n-1}\,S\frac{x^m dx}{\overline{1-x^2}^{\frac{n-1}{2}}},$$

&

& $\frac{n-m-2}{n-1} S \frac{x^m dx}{(1-x^2)^{\frac{n-1}{2}}} =$

$\frac{(n-m-2)\times x^{m+1}}{(n-1)\times(n-3)\times(1-x^2)^{\frac{n-3}{2}}} + \frac{(n-m-2)\times(n-m-4)}{(n-1)\times(n-3)}$

$S\frac{x^m dx}{(1-x^2)^{\frac{n-3}{2}}}$, atque ita semper procedendo, usurpato vicibus $h$ eodem I. Theoremate invenietur

$$S\frac{x^m dx}{(1-x^2)^{\frac{n+1}{2}}} = \frac{x^{m+1}}{(n-1)\times(1-x^2)^{\frac{n-1}{2}}} +$$

$$\frac{(n-m-2)\times x^{m+1}}{(n-1)\times(n-3)\times(1-x^2)^{\frac{n-3}{2}}} + \frac{(n-m-2)\times(n-m-4)\times x^{m+1}}{(n-1)\times(n-3)\times(n-5)\times(1-x^2)^{\frac{n-5}{2}}} +$$

$$\frac{(n-m-2)\times(n-m-4)\times(n-m-6)\times x^{m+1}}{(n-1)\times(n-3)\times(n-5)\times(n-7)\times(1-x^2)^{\frac{n-7}{2}}} + \&c.$$

$$\ldots\ldots + \frac{(n-m-2)\times(n-m-4)\times(n-m-6)\ldots\times(n-m-2h+2)\times x^{m+1}}{(n-1)\times(n-3)\times(n-5)\times(n-7)\ldots\times(n-2h+1)\times(1-x^2)^{\frac{n-2h+1}{2}}} +$$

n-m



$$\frac{(n-m-2)\times(n-m-4)\times(n-m-6)\times(n-m-8)\ldots\times(n-m-2h)}{(n-1)\times(n-3)\times(n-5)\times(n-7)\ldots\ldots\times(n-2h+1)}$$

$S\frac{x^m dx}{(1-x^2)^{\frac{n-2h+1}{2}}}$. Ex hujus æquationis inspectione statim colligitur summam inventam esse algebraicam, ubi n — m sit numerus par positivus, seu = 2h, quia tunc evanescit ultimus æquationis terminus, sive fracti sint n, & m, sive integri; adeoque in hoc casu $S\frac{d\omega \cos\omega^m}{\sin\omega^n}$ invenietur, algebraica expressa per solos sinus, & cosinus arcus $\omega$. Liquet præterea,

$S\frac{x^m dx}{(1-x^2)^{\frac{n+1}{2}}}$ esse algebraice integrabilem, ubi m sit = 0, n sit numerus par, ex: gr. = 2h; in hoc enim casu invenitur

$$S\frac{x^m dx}{(1-x^2)^{\frac{n+1}{2}}} = \frac{x}{(n-1)\times(1-x^2)^{\frac{n-1}{2}}} +$$

n — 2

$$\frac{\overline{n-2}\times x}{\overline{n-1}\times\overline{n-3}\times\overline{1-x^2}^{\frac{n-3}{2}}} + \frac{\overline{n-2}\times\overline{n-4}\times x}{\overline{n-1}\times\overline{n-3}\times\overline{n-5}\times\overline{1-x^2}^{\frac{n-5}{2}}} + \&c.$$

$$\ldots + \frac{\overline{n-2}\times\overline{n-4}\times\overline{n-6}\ldots\ldots\times 2\, x}{\overline{n-1}\times\overline{n-3}\times\overline{n-5}\times\overline{n-7}\ldots\ldots\times\overline{1-x^2}^{\frac{1}{2}}}.$$ *Si* vero

$n$ ſit numerus impar, uſurpato vicibus $\frac{n-1}{2}$ eodem 1. Theoremate invenietur eadem ſumma $= \frac{x}{\overline{n-1}\times\overline{1-x^2}^{\frac{n-1}{2}}} + \frac{\overline{n-2}\times x}{\overline{n-1}\times\overline{n-3}\times\overline{1-x^2}^{\frac{n-3}{2}}} +$

$$\frac{\overline{n-2}\times\overline{n-4}\times x}{\overline{n-1}\times\overline{n-3}\times\overline{n-5}\times\overline{1-x^2}^{\frac{n-5}{2}}} + \&c.\ldots\ldots\ldots +$$

$$\frac{\overline{n-2}\times\overline{n-4}\ldots\ldots\times 3\, x}{\overline{n-2}\times\overline{n-3}\times\overline{n-5}\ldots\ldots\times 2\times\overline{1-x^2}^{\frac{2}{2}}} +$$

$$\frac{\overline{n-2}\times\overline{n-4}\ldots\ldots\ldots\times 1}{\overline{n-1}\times\overline{n-3}\ldots\ldots\ldots\times 2}\times S\frac{dx}{1-x^2}$$

Conſtat autem ex dictis ſuperius formulam $S\frac{dx}{1-x^2} = -\,L\ \text{tang}\ \frac{1}{2}\,\omega$; hinc ſumma ſuperior ab hyperbolæ quadratura pendebit. Caſus alios data opera prætermitto, quia ex dictis



dictis ſponte fluunt, & in oculos incurrunt. Sit ergo

### PROBLEMA VII.

Formulam IV. $S\frac{x^{-m}\,dx}{\overline{1-x^2}^{\frac{n+1}{2}}}$ in hypoteſi exponentium $m$, $n$ fractorum ad integrationem revocare.

### SOLUTIO

Si $n$ ſit fractio quæcumque, modo $m$ ſit numerus integer impar, Formula

$$-S\frac{x^{-m}\,dx}{\overline{1-x^2}^{\frac{n+1}{2}}}$$ ad notas regulas revocatur, factis $m = 2f+1$, $n = \frac{h}{r}$, $\overline{1-x^2}^{\frac{1}{2r}} = y$; & prodibit $S\frac{x^{-m}\,dx}{\overline{1-x^2}^{\frac{n+1}{2}}} =$

$$\frac{r\,y^{r-h-1}\,dy}{\overline{1-y^{2r}}^{f+1}}$$, quæ, ut norunt Analyſtæ, nullum parit obſtaculum. Pro aliis vero caſibus

sibus, usurpato vicibus $h$ de more eodem I. Theoremate, eruetur $S\frac{x^{-m}dx}{(1-x^2)^{\frac{n+1}{2}}}$

(positivam considero commodi gratia) =

$$\frac{1}{(n-1)\times x^{m-1}\times(1-x^2)^{\frac{n-1}{2}}}+\frac{n+m-2}{(n-1)\times(n-3)\times x^{m-1}\times(1-x^2)^{\frac{n-3}{2}}}+$$

$$\frac{(n+m-2)\times(n+m-4)}{(n-1)\times(n-3)\times(n-5)\times x^{m-1}\times(1-x^2)^{\frac{n-5}{2}}}+\&c.\dots+$$

$$\frac{(n+m-2)\times(n+m-4)\times(n+m-6)\dots\dots\times(n+m-2h+2)}{(n-1)\times(n-3)\times(n-5)\times(n-7)\dots\times(n-2h+1)\times x^{m-1}\times(1-x^2)^{\frac{n-2h+1}{2}}}$$

$$+\frac{(n+m-2)\times(n+m-4)\times(n+m-6)\dots\times(n+m-2h)}{(n-1)\times(n-3)\times(n-5)\dots\dots\times(n-2h+1)}$$

$S\frac{x^{-m}dx}{(1-x^2)^{\frac{n-2h+1}{2}}}$. Ex hac aequatione statim apparet Formulam $S\frac{x^{-m}dx}{(1-x^2)^{\frac{n+1}{2}}}$ summam reci-



recipere algebraicam, ubi $m$, & $n$ sint numeri pares, & $m+n=2h$, sive fracti sint, sive integri exponentes $m$, $n$; ac post numerum operationum $\frac{m+n}{2}$ prodibit

$$S\frac{x^{-m}dx}{(1-x^2)^{\frac{n+1}{2}}}=\frac{1}{(n-1)\times x^{m-1}\times(1-x^2)^{\frac{n-1}{2}}}+$$

$$\frac{n+m-2}{(n-1)\times(n-3)\times x^{m-1}\times(1-x^2)^{\frac{n-3}{2}}}+\&c.\dots\dots$$

$$+\frac{(n+m-2)\times(n+m-4)\dots\dots\times 2}{(n-1)\times(n-3)\times(n-5)\dots\dots\times(1-m)\times x^{m-1}\times(1-x^2)^{\frac{1-m}{2}}}$$

At si $m+n=$ pari, sed ambo impares, tum subrogando $\frac{n-1}{2}$ pro $h$ orietur

$$S\frac{x^{-m}dx}{(1-x^2)^{\frac{n+1}{2}}}=\frac{1}{(n-1)\times x^{m-1}\times(1-x^2)^{\frac{n-1}{2}}}$$

$$+\frac{n+m-2}{(n-1)\times(n-3)\times x^{m-1}\times(1-x^2)^{\frac{n-3}{2}}}+\&c.\dots\dots+$$

$$\frac{(n+m-2)\times(n+m-4)\dots\dots\times(m+3)}{(n-1)\times(n-3)\times(n-5)\dots\dots\times 2\times x^{m-1}\times(1-x^2)}+$$



n+m

$$\frac{\overline{n+m-2}\times\overline{n+m-4}\ldots\ldots\ldots\times\overline{m+1}}{\overline{n-1}\times\overline{n-3}\ldots\ldots\ldots\ldots\times 2}\times$$

$$\int\frac{x^{-m}dx}{1-x^2}$$ , adeoque in hoc caſu ſumma

erit partim algebraica , partim logarithmi-

ca , quia $\int\frac{dx}{x^{m}\times\overline{1-x^2}}$ pendet ab hyper-

bolæ quadratura , ut omnes norunt . Ex hactenus dictis palam fit, quod ſi n ſit $=0$,

m numerus par , erit $\int\frac{dx}{x^m\times\overline{1-x^2}^{\frac{1}{2}}}=$

$$-\frac{\overline{1-x^2}^{\frac{1}{2}}}{\overline{m-1}\,x^{m-1}}+\frac{\overline{m-2}\times\overline{1-x^2}^{\frac{3}{2}}}{3x^{m-1}}$$

$$-\frac{\overline{m-2}\times\overline{m-4}\times\overline{1-x^2}^{\frac{5}{2}}}{3\times5\times x^{m-1}}+\&c.\ldots$$

$$\frac{\overline{m-2}\times\overline{m-4}\times\overline{m-6}\ldots\times2\times\overline{1-x^2}^{\frac{m-1}{2}}}{3\times5\times7\ldots\times\overline{m-1}\times x^{m-1}}$$ . Si

vero



vero m ſit numerus impar , ex: gr: $=2h+1$ , tum expeditiſſima erit formulæ

$\int\frac{dx}{x^{2h+1}\times\overline{1-x^2}^{\frac{1}{2}}}$ integratio; facta enim ſub-

ſtitutione $1-x^2=u^2$ , mutabitur in

$\int\frac{-du}{\overline{1-u^2}^{h+1}}$ nullo negotio integrabilem ;

ac pendentem ab $\int\frac{du}{1-u^2}$ , ſcilicet a L

tang $(45^\circ+\frac{1}{2}\omega)$ , ut in Eulerianarum formularum integratione invenimus. Caſus alios evolvere ſuperſedeo , quia ex dictis facile eruuntur , & profluunt . Supereſſet modo,

integranda Formula III. $\int\frac{x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$ ;

at nullus uſque adhuc mihi ſe ſe obtulit caſus , quo , ſuppoſitis m , & n numeris fractis , Formula evadat integrabilis . Reapſe uſurpato I. Theoremate vicibus h prodit

$$\int\frac{x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}}=-\frac{x^{m+1}\times\overline{1-x^2}^{\frac{n+1}{2}}}{n+1}$$

$$-\,.\,\frac{\overline{n+m+2}\times x^{m+1}\times\overline{1-x^2}^{\frac{n+3}{2}}}{\overline{n+1}\times\overline{n+3}}$$

$$-\,.\,\frac{\overline{n+m+2}\times\overline{n+m+4}\times x^{m+1}\times\overline{1-x^2}^{\frac{n+5}{2}}}{\overline{n+1}\times\overline{n+3}\times\overline{n+5}}\ -\ \&c.\ldots$$

$$-\,.\,\frac{\overline{n+m+2}\times\overline{n+m+4}\ldots\times\overline{n+m+2h-2}\times x^{m+1}\times\overline{1-x^2}^{\frac{n+2h-1}{2}}}{\overline{n+1}\times\overline{n+3}\times\overline{n+5}\ldots\ldots\times\overline{n+2h-1}}\ +$$

$$\frac{\overline{n+m+2}\times\overline{n+m+4}\ \ldots\ldots\ldots\times\overline{n+m+2h}}{\overline{n+1}\times\overline{n+3}\ \ldots\ldots\ldots\ldots\times\overline{n+2h-1}}$$

$S\frac{x^m dx}{\overline{1-x^2}^{\frac{-n-2h+1}{2}}}$. Porro hæc æquatio nusquam interrumpitur, ut algebraica fiat, pendetque semper ab $S\frac{x^m dx}{\overline{1-x^2}^{\frac{-n-2h+1}{2}}}$, quæ similis est Formulæ propositæ $S\frac{x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$;

adeoque nullus occurrit casus, quo possit eadem Formula ad integrationem revocari in



in hypotesi quod exponentes m, n fracti sint. At si eorum alteruter fractus assumatur, tum haudquaquam desperandum erit de Formulæ integratione. Esto itaque

## PROBLEMA VIII.

Formulam III. $S - x^m dx\times\overline{1-x^2}^{\frac{n-1}{2}}$

in hypothesi unius exponentis fracti ad integrationem redigere.

## SOLUTIO

Si m sit numerus fractus quicumque, n vero sit numerus positivus impar, Formula proposita protinus integratur elevando ad potestatem integram $\frac{n-1}{2}$ quantitatem $1-x^2$ sub vinculo comprehensam. Si vero n sit numerus fractus, tum Formula

$$S - x^m dx\times\overline{1-x^2}^{\frac{n-1}{2}},\ \text{seu}\ S\frac{-x^m dx}{\overline{1-x^2}^{\frac{1-n}{2}}}$$

 per

per substitutionem $\overline{1-x^2}^{\frac{1-n}{2}} = u$, degenerabit in $S\, u^{\frac{n+1}{1-n}} du \times \overline{1-u^{\frac{2}{1-n}}}^{\frac{m-1}{2}}$.

Jam vero $S\, u^{\frac{n+1}{1-n}} du \times \overline{1-u^{\frac{2}{1-n}}}^{\frac{m-1}{2}}$ integrabilis est, si m sit numerus impar positivus; nam elevata quantitate $1-u^{\frac{2}{1-n}}$ ad potestatem integram $\frac{m-1}{2}$ fit protinus integratio. Si m sit numerus impar, sed negativus, formula superior erit

$$S \frac{u^{\frac{n+1}{1-n}} du}{\overline{1-u^{\frac{2}{1-n}}}^{\frac{-m+1}{2}}}$$, & quantitas $\overline{1-u^{\frac{2}{1-n}}}^{\frac{m+1}{2}}$

erit potestas integra; proindeque

$$S \frac{u^{\frac{n+1}{1-n}} du}{\overline{1-u^{\frac{2}{1-n}}}^{\frac{m+1}{2}}}$$ pendebit, ut Analystæ com-





pertum habent, ab $S \frac{u^{\frac{n+1}{1-n}} du}{1-u^{\frac{2}{1-n}}}$.

Porro $S \frac{u^{\frac{n+1}{1-n}} du}{1-u^{\frac{2}{1-n}}}$ etiam in hypothesi n fracti ad notas regulas statim revocatur, factis $\frac{n+1}{1-n} = \frac{h}{l}$, & $\frac{2}{1-n} = \frac{p}{q}$, ut sit

$$S \frac{u^{\frac{n+1}{1-n}} du}{1-u^{\frac{2}{1-n}}} = S \frac{u^{\frac{h}{l}} du}{1-u^{\frac{p}{q}}}$$, designantibus h, l, p, q numeris integris. Fiat $u = z^{lq}$, eritque $S \frac{u^{\frac{h}{l}} du}{1-u^{\frac{p}{q}}} =$

$$S \frac{lq z^{hq+lq-1} dz}{1-z^{lp}}$$, quæ inventis de more factoribus denominatoris $1-z^{lp}$ integratur per notas regulas, penditque a circuli, & hyperbolæ quadratura, ut notum jam est.

Vix manum de tabula posueram, cum ecce offert se mihi nec opinanti casus singularis; quo proposita Formula

$\int \overline{-x}^{m} dx \times \overline{1-x^2}^{\frac{n-1}{2}}$ per notas quadraturas integratur, licet exponentes ambo m, n fracti assumantur. Si enim exponentes m, n tales fractiones sint, ut $\frac{-m-n}{2}-1$ evadat numerus integer negativus, quod potest frequentissime usuvenire, tum $\int \overline{-x}^{m} dx \times \overline{1-x^2}^{\frac{n-1}{2}}$ pendebit a notis circuli, & hyperbolæ quadraturis. Immo si exponentes m, n sint hujusmodi fractiones, ut $\frac{-m-n}{2}-1$ sit numerus integer affirmativus (quod potest etiam sæpissime contingere), tum $\int \overline{-x}^{m} dx \times \overline{1-x^2}^{\frac{n-1}{2}}$ algebraice integrabilis deprehendetur. Ut duo hæc luculentissime ostendantur, esto $1-x^2 = x^2 z$, adeoque $x = \frac{1}{\sqrt{1+z}}$; hac facta substitutione in Formula proposita $\int \overline{-x}^{m} dx \times \overline{1-x^2}^{\frac{n-1}{2}}$, fiet ipsa =

S ½



$\int \frac{1}{2} z^{\frac{n-1}{2}} dz \times \overline{1+z}^{\frac{-m-n}{2}-1}$. Jam vero si $\frac{-m-n}{2}-1$ sit numerus integer negativus, invenietur per notas regulas $\int \frac{1}{2} z^{\frac{n-1}{2}} dz \times \overline{1+z}^{\frac{-m-n}{2}-1}$ pendere ab $\int \frac{z^{\frac{n-1}{2}} dz}{1+z}$, sive (facto $\frac{n-1}{2} = \frac{p}{r}$) ab $\int \frac{z^{\frac{p}{r}} dz}{1+z}$.

Fiat modo $z = y^r$, eritque $\int \frac{z^{\frac{p}{r}} dz}{1+z} = \int \frac{r y^{p+r-1} dy}{1+y^r} = \int \left( r y^{p-1} dy - \frac{r y^{p-1} dy}{1+y^r} \right) = \frac{r}{p} y^p - \int \frac{r y^{p-1} dy}{1+y^r}$. Porro Formula $\int \frac{r y^{p-1} dy}{1+y^r}$ integratur, ut constat, per notas circuli, & hyperbolæ quadraturas, resoluto de more denominatore $1+y^r$ in factores suos per notissimas binomii regulas.

Ergo

Ergo proposita Formula $\int x^{m} dx \times \overline{1 - x^{2}}^{\frac{n-1}{2}}$ a circuli, & hyperbolæ quadraturis pendebit, ubi exponentes m, n tales fractiones sint, ut $\frac{-m-n}{2} - 1$ sit numerus integer negativus; ex: gr: si m sit $= \frac{3}{2}$, n $= \frac{1}{2}$; m $= \frac{3}{2}$, n $= \frac{5}{2}$; m $= \frac{7}{2}$, n $= \frac{1}{2}$; m $= \frac{9}{2}$, n $= \frac{3}{2}$ &c. in infinitum. Si vero $\frac{-m-n}{2} - 1$ sit numerus integer positivus, vel etiam nihilum, licet m, n fracti sint, ut si ex gr. m $= -\frac{1}{2}$, n $= -\frac{5}{2}$; m $= -\frac{3}{2}$, n $= -\frac{1}{2}$; m $= -\frac{9}{2}$; n $= -\frac{3}{2}$; m $= -\frac{7}{2}$, n $= -\frac{9}{2}$ &c. in infinitum; tunc prodibit summa algebraica, quia elevata quantitate $1 + z$ ad potestatem integram positivam $\frac{-m-n}{2} - 1$, fiet protinus formulæ $\int \frac{1}{2} z^{\frac{n-1}{2}} dz \times \overline{1 + z}^{\frac{-m-n}{2} - 1}$ integratio, ut per se patet.





# OPUSCULUM II.

## DE THEOREMATE ROGERII COTES, ejus usu, utilitate, præstantia.

Geometrarum neminem ignorare arbitror, quanta sit Cotesiani Theorematis utilitas, ac pene necessitas in Analysi Sublimiori, in Calculo potissimum Integrali, ubi differentiales Formulæ binomiam, vel trinomiam quantitatem in denominatore involventes integrandæ proponuntur, quæ hujus Theorematis ope facillime extricantur. Geometris illud primum innotuit, postquam anno 1722 posthumum Opus acutissimi Geometræ Rogerii Cotes inscriptum *Harmonia mensurarum, sive Analysis, & Synthesis per Rationum, & Angulorum mensuras promotæ* opera Cel. Roberti Smith in lucem prodivit. In eximio hoc opere primum apparuit pulcherrimum istud, elegantissimumque Theorema, sed absque ulla demonstratione propositum, quam *altæ indaginis* appellat Jacobus Hermannus in *Supplemento circa Problema a Tayloro propositum* tom. VI. *Comment. Petropolit.* Ex hoc Theoremate prodiit veluti ex trunco surculus Formulæ illius celeberrimæ constructio, ad quam Geometra eximius Taylorus Mathematicos non Anglos provocaverat, Formulæ scilicet

$$\frac{z^{\frac{\delta}{\lambda}n-1}\,dz}{e+fz^{n}+gz^{2n}}$$, quæ illius ope expeditissimam reductionem admittit, & denominator in factores secundi gradus maximo compendio, ac veluti unico calami ductu resolvitur. Formulæ hujus Taylorianæ constructionem, ad quam omnes certatim Geometræ extra Angliam degentes convolaverant, invenit etiam duplici methodo summa analyseos concinnitate, & elegantia absque ullo Cotesiani Theorematis subsidio Geometra ex nostris percelebris, nullique secundus Marchio Julius Fagnanus in egregio Opere *Produzioni Matematiche* tom. II. pag. 293. Verum ob maximam, quæ inter Cotesianum Theorema, & Problema Taylorianum intercedit, affinitatem, & cognationem, quum hoc illius Corollarium sit a quolibet vel mediocri Geometra deducendum, Cl. Hermannus loco citato affirmare non dubitat, *Problema illud Taylorianum non præclarissimum Taylorum ipsum, sed acutissimum, dum viveret, Rogerium Cotesium auctorem habuisse*. Quidquid tamen hac de re sit, illud certo constat, Taylorianam Formulam, quæ Integralem Calculum adeo amplificavit, ejusque fines tantopere produxit, per Cotesianum Theorema facillime construi, & ad integrationem revocari. Illud unice exoptandum supe-



perat, ut Theorematis Auctor perspicacissimus illius demonstrationem publici juris fecisset, quæ tamen, ut dictum est, in posthumo ipsius Opere desideratur. Restituit illam quidem, supplevitque cel. Robertus Smith, quæ ad calcem Operis Cotesiani adjecta conspicitur, sed Geometrarum non videtur obtinuisse suffragia, quia ambagibus involuta perspicuitate destituitur. Eandem dederat etiam Cl. Pembertonus in *Epistola ad Amicum de Cotesii inventis* Cotesiano operi adjecta in editione Londinensi anni 1722.; verum pluribus ea nominibus carpitur ab acutissimo Jo: Bernoullio *Operum tom.* 4. his verbis: *Exhibuit postea aliquam Pembertonus, eidem libro Cotesiano insertam, sed tam longam, tam intricatam, ut tædiosum sit examinare, utrum omnia recte se habeant, nullusque lateat paralogismus.* Hinc factum est, ut illustriores Geometræ in hac demonstratione invenienda magna animorum contentione allaborarent, & ut quisque felicior suam proponeret, & Litterario Orbi dijudicandam relinqueret. Joannes Bernoullius eandem Mathematicorum Reipublicæ impertivit Operum suorum *tom. IV. N.° CLX.*, Jacobus Hermannus in Commentariis Petropolitanis loco citato, Abrahamus De Moivre in eximio Opere inscripto *Miscellanea Analytica De Seriebus, & Quadraturis lib. II. Cap. I.*, Samuel Koenigius in *Novis Actis Eruditorum Lips. ann.* 1741. *Mens. Januar.*, Carolus Wal-

Walmesleyus Benedictinus Anglus in elaboratissimo Opere *Analyse des Mesures des Rapports, & des Angles*, &c. & novissime Bougainvillius in Opere egregio *Traité Du Calcul. Integral tom. I. Introduction § LXIII.* Verum demonstrationes hujusmodi, quæ a tam præclaris Viris diversis temporibus excogitatæ fuerunt, ut ut ingeniosissimæ, & acutissimæ ad Tyronum captum parum accommodatæ videntur, propterea quod vel nimia prolixitate Juvenum ingenia obruunt, vel nimia brevitate torquent. Inter abstrusiores hasce demonstrationes principem certe locum obtinere videtur cum perspicuitate, tum elegantia demonstratio acutissimi Abrahami De Moivre; attamen perspicacissimus Jo: Bernoullius de eadem loco citato ita edisserit „ Felicioribus „ vero auspiciis rem aggressus Clarissimus Moi„ vræus, vir profunde doctus atque in ana„ lyticis versatissimus, dedit hujus Theore„ matis eruendi viam expeditissimam, & qui„ dem ad alia hujus generis longissime se por„ rigentem, quousque impar Adversarius pe„ netrare non poterat. Methodus Moivræana „ petita est ex natura serierum, ut vocant, „ recurrentium, quæ licet in obscurissimis dis„ quisitionibus plerumque magno sint auxilio, „ nonnullis tamen ista operandi methodus non „ satis naturalis esse videtur „; & Geometra si quis alius subtilis, & peracutus. Jacobus Hermannus de eadem loco citato verba faciens diserte ait: *In erudito hoc opere* (Moivræano) *occurrit pulchrum theorema de divisione Circuli in quotcumque partes æquales, ex quo deinceps magna brevitate & concinnitate deducit tum demonstrationem theorematis Cotesiani, de quo supra, tum etiam fractiones construendas ad formas simpliciores. Verumtamen ..... post lectionem utlibet attentam corollariorum usum Lemmatis illustrantium, semper aliquis scrupulus remansit, impediens quo minus credere possem Acutissimi Viri mentem me recte percepisse.* Neque tamen Hermanni ipsius demonstratio clarior, aut facilior Moivreana videtur, quod ceteroquin sperare fas erat ab eo, qui Moivreanam obscuritatis insimulat; junioribus enim Geometris non unum parit incommodum tum nimia computationum, quas utplurimum omittit, brevitate, ne indicatis quidem Analyseos vestigiis, tum quia abstrusiori utitur methodo, & per salebrosas progreditur vias. Itaque operæ pretium me facturum existimavi, si novam, eamque faciliorem Cotesiani Theorematis demonstrationem Analyseos sublimioris studiosis proponerem, & ad fæcundissimum hoc Geometriæ inventum promptumque, expeditumque aditum aperirem. Novos quosdam usus patefeci radicum unitatis positivæ, & negativæ, ex quibus deinceps Theorematis demonstrationem deduxi. Adjeci usus amplissimos Theorematis ejusdem in Analysi sublimiori, omniaque summa qua potui brevitate & perspicuitate demonstrare curavi, ne quid Junioribus Geometris desiderandum relinqueretur; sed plana omnia, & patefacta invenirent.

### PROBLEMA I.

Invenire omnes radices unitatis, quæ sint gradus $n$.

### SOLUTIO.

Omnes radices unitatis gradus $n$ æquales sunt radicibus æquationis $x^n - 1 = 0$, quia in hac æquatione omnes ipsius $x$ valores evecti ad potentiam $n$ æquales sunt unitati; & quoniam valores hujusmodi sunt numero $= n$, totidem idcirco erunt unitatis ipsius radices. Porro æquatio $x^n - 1 = 0$ vel est gradus paris, vel imparis; si $n$ est numerus par, duas tantum radices reales continebit, nimirum $+1$, & $-1$, reliquæ omnes erunt imaginariæ, quia numerus alius quicumque ab unitate diversus vel major erit unitate, vel minor, & evectus ad potestatem $n$ major rursus erit, vel minor unitate, quod fieri non potest; si vero $n$ sit numerus impar, æquatio superior unam tantum realem radicem habebit $+1$, & ceteras omnes imaginarias ob rationem jam indicatam. Itaque inter radices unitatis, quæ sint gradus $n$, inerit semper numerus par radicum imaginariarum. Harum quælibet exprimi potest per $a + b\sqrt{-1}$, sumptis $a$, & $b$ pro numeris realibus sive positivis, sive negativis, acceptoque etiam $a$ pro nihilo, ut sub hac forma radices omnes possibiles imaginariæ comprehendantur (a). Harum radicum dimidium non differt ab altero dimidio nisi signo, quod quantitati imaginariæ præfigitur; nimirum si dimidium harum radicum sit $a + b\sqrt{-1}$, $e + f\sqrt{-1}$, $d + h\sqrt{-1}$ &c. dimidium alterum erit $a - b\sqrt{-1}$, $e - f\sqrt{-1}$, $d - h\sqrt{-1}$ &c.; cujus ratio hæc est: Quoniam ex hypothesi $a + b\sqrt{-1}$ est radix æquationis $x^n - 1 = 0$, erit $\overline{a + b\sqrt{-1}}^{\,n} = 1$, ergo

$$a^n + na^{n-1}b\sqrt{-1} - \frac{n\times\overline{n-1}}{2}a^{n-2}b^2 - \frac{n\times\overline{n-1}\times\overline{n-2}}{2\times 3}a^{n-3}b^3\sqrt{-1} + \frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times 3\times 4}a^{n-4}b^4 \ldots \pm$$

 $b^n$

(a) *Radices quascumque æquationum imaginarias per simplicissimam formulam* $A \pm B\sqrt{-1}$ *designari posse ingeniosissime demonstravit Alembertius in Monum. Berolin. tom. II.* Recherches sur le Calcul Integral *art.* XI., *Eulerus eorumdem Monum. tomo* V. Recherches sur les Racines imaginaires des equations §. 76; *Bougainvillius* Traité du Calcul Integral *tom.* I. *Introduct. art.* LXVII.

$b^n$ (si n est par), vel $\pm b^n \sqrt{-1}$ (si n est impar) $= 1$. In hac æquatione termini omnes imaginarii se se invicem destruent, & reales omnes simul sumpti erunt $= 1$, adeoque

$$n a^{n-1} - \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3} a^{n-3} b^2 \ldots\ldots =$$

$$0, \text{ vel } - n a^{n-1} + \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3} a^{n-3} b^3$$

$$\ldots\ldots = 0; \ \& \ a^n - \frac{n \times \overline{n-1}}{2} a^{n-2} b^2 +$$

$$\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4} a^{n-4} b^4 \ldots\ldots = 1;$$

ergo etiam $a^n - n a^{n-1} b \sqrt{-1} -$

$$\frac{n \times \overline{n-1}}{2} a^{n-2} b^2 + \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3} a^{n-3} b^3 \sqrt{-1} +$$

$$\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4} a^{n-4} b^4 \ldots\ldots \pm b^n \text{ (si n est}$$

par), vel $\mp b^n \sqrt{-1}$ (si n est impar) $= 1$;

est



est autem primum hujus æquationis membrum $= \overline{a - b\sqrt{-1}}^n$, ut per se patet; ergo

$\overline{a - b\sqrt{-1}}^n = 1$, proindeque $a - b\sqrt{-1}$ erit una ex radicibus gradus n unitatis. Eadem de $e - f\sqrt{-1}$, $d - h\sqrt{-1}$ &c. recurrit ratiocinatio. Itaque radices omnes æquationis $x^n - 1 = 0$, si n est numerus par, sunt $\pm 1$, $a \pm b\sqrt{-1}$, $e \pm f\sqrt{-1}$, $d \pm h\sqrt{-1}$ &c.; si vero n est impar, erunt $+1$, $g \pm k\sqrt{-1}$, $p \pm q\sqrt{-1}$ &c.; ac proinde si in hypothesi n paris dividatur $x^n - 1 = 0$ per

$\overline{x-1} \times \overline{x+1} = 0$, seu per $x^2 - 1 = 0$, æquatio per divisionem orta $x^{n-2} + x^{n-4} + x^{n-6}$

$\ldots\ldots + 1 = 0$ habebit radices omnes imaginarias, scilicet $a \pm b\sqrt{-1}$, $e \pm f\sqrt{-1}$, $d \pm h\sqrt{-1}$ &c.; & si in hypothesi n imparis dividatur $x^n - 1 = 0$ per $x - 1 = 0$, æquatio inde resultans $x^{n-1} + x^{n-2} + x^{n-3}$

$\ldots\ldots + x + 1 = 0$ radices tantum imaginarias complectetur, videlicet $g \pm k\sqrt{-1}$, $p \pm q\sqrt{-1}$ &c. Jam vero si in priori æquatione pro x subrogetur $\frac{1}{y}$, fiet $y^{n-2} + y^{n-4}$

 $+ y^{n-6}$

$\pm y^{n-6}$ ....... † $1 = 0$, cujus radices erunt pariter $a \pm b\sqrt{-1}$, $e \pm f\sqrt{-1}$ &c.; & si in æquatione altera eadem fiat substitutio, prodibit $y^{n-1} \pm y^{n-2} \pm y^{n-3}$ ........ † $y$ + $1 = 0$, cujus radices erunt rursus $g \pm K\sqrt{-1}$, $p \pm q\sqrt{-1}$ &c. Est autem $x = \frac{1}{y}$; ergo omnes valores ipsius x orientur dividendo unitatem per valores y, seu per valores alios ejusdem x, proindeque valores ipsius x erunt $\frac{1}{a \pm b\sqrt{-1}}$, $\frac{1}{a - b\sqrt{-1}}$, $\frac{1}{e \pm f\sqrt{-1}}$, $\frac{1}{e - f\sqrt{-1}}$, $\frac{1}{d \pm h\sqrt{-1}}$, $\frac{1}{d - h\sqrt{-1}}$ &c. & $\frac{1}{g \pm K\sqrt{-1}}$, $\frac{1}{g - K\sqrt{-1}}$, $\frac{1}{p \pm q\sqrt{-1}}$, $\frac{1}{p - q\sqrt{-1}}$ &c., seu $\frac{a - b\sqrt{-1}}{a^2 \pm b^2}$, $\frac{a \pm b\sqrt{-1}}{a^2 \pm b^2}$, $\frac{e - f\sqrt{-1}}{e^2 \pm f^2}$, $\frac{e \pm f\sqrt{-1}}{e^2 \pm f^2}$, $\frac{d - h\sqrt{-1}}{d^2 \pm h^2}$, $\frac{d \pm h\sqrt{-1}}{d^2 \pm h^2}$ &c. & $\frac{g - K\sqrt{-1}}{g^2 \pm K^2}$, $\frac{g \pm K\sqrt{-1}}{g^2 \pm K^2}$, $\frac{p - q\sqrt{-1}}{p^2 \pm q^2}$, $\frac{p \pm q\sqrt{-1}}{p^2 \pm q^2}$ &c.; at valores omnes ipsius x evecti ad potestatem n sunt unitati æquales; igitur

a $\pm$



$\frac{\overline{a \mp b\sqrt{-1}}^n}{\overline{a^2 \pm b^2}^n}$, $\frac{\overline{e \mp f\sqrt{-1}}^n}{\overline{e^2 \pm f^2}^n}$, $\frac{\overline{d \mp h\sqrt{-1}}^n}{\overline{d^2 \pm h^2}^n}$ &c. & $\frac{\overline{g \mp K\sqrt{-1}}^n}{\overline{g^2 \pm K^2}^n}$, $\frac{\overline{p \mp q\sqrt{-1}}^n}{\overline{p^2 \pm q^2}^n}$ &c. erunt unitati æquales; sed ex jam demonstratis $\overline{a \mp b\sqrt{-1}}^n$, $\overline{e \mp f\sqrt{-1}}^n$, $\overline{d \mp h\sqrt{-1}}^n$ &c. & $\overline{g \mp K\sqrt{-1}}^n$, $\overline{p \mp q\sqrt{-1}}^n$ &c. unitati æquantur; ergo etiam $\overline{a^2 \pm b^2}^n$, $\overline{e^2 \pm f^2}^n$, $\overline{d^2 \pm h^2}^n$ &c. & $\overline{g^2 \pm K^2}^n$, $\overline{p^2 \pm q^2}^n$, &c. unitati æquales sunt, adeoque etiam $a^2$ † $b^2$, $e^2$ † $f^2$, $d^2$ † $h^2$ &c., & $g^2 + K$, $p^2$ † $q^2$ &c. unitati æquabuntur; & quia a, b, e, f &c. &c. sunt reales quantitates, erit earum una quælibet unitate minor. Itaque radices omnes gradus n unitatis, seu radices æquationis $x^n - 1 = 0$, erunt, si n est par, $\pm 1$, $a \pm b\sqrt{-1}$, $e \pm f\sqrt{-1}$, &c.; si n est impar, † 1, $g \pm K\sqrt{-1}$, $p \pm q\sqrt{-1}$, &c. Q. E. I.



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Corollarium I. Productum ex binis radicibus imaginariis $\overline{a + b\sqrt{-1}} \times \overline{a - b\sqrt{-1}}$ &c. $\overline{g + K\sqrt{-1}} \times \overline{g - K\sqrt{-1}}$ &c. unitatem facit.

Corollarium II. Dati binomii $x^n - 1$ diviſores ſimplices ſunt $x - 1$, $x + 1$, $x - a - b\sqrt{-1}$, $x - a + b\sqrt{-1}$, $x - e - f\sqrt{-1}$, $x - e + f\sqrt{-1}$ &c., ſi n ſit numerus par, ac totidem erunt, quot unitatibus conſtat n, proindeque omnes in ſe invicem ducti producent factum $x^n - 1$; & quoniam par eſt diviſorum numerus, omnium productum habebitur ducendo ſemper in ſe invicem binos proximos, quo facto prodibit $x^n - 1 = \overline{x^2 - 1} \times \overline{x^2 - 2ax + 1} \times \overline{x^2 - 2ex + 1} \times$ &c., atque ita reſolvi ſemper poterit binomium $x^n - 1$ in diviſores reales ſecundi gradus, quorum numerus erit $\frac{n}{2}$. Atque hinc infertur eſſe $1 - x^n = \overline{1 - x^2} \times \overline{x^2 - 2ax + 1} \times \overline{x^2 - 2ex + 1} \times$ &c. Si vero n ſit numerus impar, diviſores ſimplices binomii $x^n - 1$ erunt $x - 1$, $x - g - K\sqrt{-1}$, $x - g + K\sqrt{-1}$, $x - p - q\sqrt{-1}$, $x - p + q\sqrt{-1}$ &c., eritque, inſtituta operatione ut ſupra, $x^n - 1 = \overline{x - 1} \times \overline{x^2 - 2gx + 1} \times \overline{x^2 - 2px + 1} \times$ &c.; quocirca binomium $x^n - 1$ in hoc caſu in diviſores reales ſecundi gradus, quorum numerus æqualis erit $\frac{n-1}{2}$, & diviſorem unum ſimplicem reſolubile erit; & $1 - x^n = \overline{1 - x} \times \overline{x^2 - 2gx + 1} \times \overline{x^2 - 2px + 1} \times$ &c.

Co-



Coroll. III. Quoniam ex demonſtratis $\overline{a \pm b\sqrt{-1}}^n$, $\overline{e \pm f\sqrt{-1}}^n$ &c. & $\overline{g \pm K\sqrt{-1}}^n$, $\overline{p \pm q\sqrt{-1}}^n$ &c. ſunt unitati æquales, etiam $\overline{a \pm b\sqrt{-1}}^{2n}$, $\overline{e \pm f\sqrt{-1}}^{2n}$, &c., & $\overline{g \pm K\sqrt{-1}}^{2n}$, $\overline{p \pm q\sqrt{-1}}^{2n}$ &c. unitati æquabuntur; ac proinde eædem $a \pm b\sqrt{-1}$, $e \pm f\sqrt{-1}$ &c., & $g \pm K\sqrt{-1}$, $p \pm q\sqrt{-1}$ &c. erunt etiam radices unitatis gradus $2n$; ergo æquatio $x^{2n} - 1 = 0$ complectetur radices omnes æquationis $x^n - 1 = 0$ præter has alias, quæ eadem ratione, qua ſupra uſi ſumus, determinantur, ſcilicet $\mu \pm \nu\sqrt{-1}$, $\lambda \pm \Delta\sqrt{-1}$, &c. vel $\phi \pm \delta\sqrt{-1}$, $\omega \pm \Theta\sqrt{-1}$ &c.; hinc quoniam radices æquationis $x^n - 1 = 0$, quando $n$ eſt par, ſunt $\pm 1$, $a \pm b\sqrt{-1}$, $e \pm f\sqrt{-1}$ &c., & facto $n$ impari ſunt $+1$, $g \pm K\sqrt{-1}$, $p \pm q\sqrt{-1}$ &c., propterea prodibit, ſuppoſito n pari, $x^{2n} - 1 =$

$$\overline{x^2 - 1} \times \overline{x^2 - 2ax + 1} \times \overline{x^2 - 2ex + 1} \times \text{ \&c.}$$

$$\times \overline{x^2 - 2\mu x + 1} \times \overline{x^2 - \lambda x + 1} \times \text{ \&c.},$$ & ſi $n$ eſt impar, erit

$x^2$

$x^{2n}-1 = \overline{x-1}\times\overline{x^2-2\beta x+1}\times\overline{x^2-2px+1}\times$ &c.

$\times\overline{x^2-2\phi x+1}\times\overline{x^2-2\omega x+1}\times$ &c.

Coroll. IV. $x^{2n}-1 = \overline{x^n-1}\times\overline{x^n+1}$; adeoque, facto n pari, erit $\overline{x^n-1}\times\overline{x^n+1} = \overline{x^2-1}\times\overline{x^2-2\alpha x+1}$ $\times\overline{x^2-2\epsilon x+1}$ &c. $\times\overline{x^2-2\mu x+1}\times\overline{x^2-2\lambda x+1}\times$ &c. unde eruitur $x^n+1 = \overline{x^2-2\mu x+1}\times\overline{x^2-2\lambda x+1}\times$ &c. in hypothesi n paris. Eadem ratione in hypothesi n imparis invenietur

$x^n+1 = \overline{x^2-2\phi x+1}\times\overline{x^2-2\omega x+1}\times$ &c., & quia in hac hypothesi inter radices unitatis negativae, adest radix negativa realis $-1$, idcirco $x^n+1 = \overline{x+1}\times\overline{x^2-2\phi x+1}\times\overline{x^2-2\omega x+1}\times$ &c.

## PROBLEMA II.

Data aequatione $l+\sqrt{l^2-1} = \overline{x+\sqrt{x^2-1}}^{\,n}$, valores omnes ipsius x invenire.

SO-



## SOLUTIO

Radices omnes cujuscumque gradus datae quantitatis aequales sunt uni ipsius quantitatis radici ductae in radices omnes unitatis positivae, si quantitas est positiva, vel in radices omnes unitatis negativae, si quantitas est negativa, dummodo ipsius unitatis radices ejusdem gradus accipiantur ac radices datae quantitatis. Esto quantitas $\pm a^n$. Quoniam $\pm a^n = a^n\times\pm 1$; erunt radices omnes quantitatis $\pm a^n$ aequales simplici radici a ductae in radices omnes ipsius $\pm 1$ gradus ejusdem n. His constitutis facile ostenditur, valores omnes ipsius x erutos ex aequatione $l+\sqrt{l^2-1} = \overline{x+\sqrt{x^2-1}}^{\,n}$ esse numero n, & reales, neque majores unitate, quando l non excedit $\pm 1$. Reapse educta ex utroque aequationis membro radice gradus n, prodibit $x+\sqrt{x^2-1} = \nu\times\overline{l+\sqrt{l^2-1}}^{\,\frac{1}{n}}$, ubi $\nu$ indicat radices omnes unitatis gradus n, unde colligitur $x = \frac{1}{2}\nu\times\overline{l+\sqrt{l^2-1}}^{\,\frac{1}{n}} + \frac{1}{2\nu}\times\overline{l+\sqrt{l^2-1}}^{\,-\frac{1}{n}}$, proindeque valores omnes ipsius

ipsius x sunt numero n, quia totidem sunt valores $v$. Esto praeterea l quantitas affirmativa, sed minor unitate, eritque $\sqrt{l^2-1}$ quantitas imaginaria $= \sqrt{1-l^2} \times \sqrt{-1}$, evectaque quantitate $l + \sqrt{1-l^2} \times \sqrt{-1}$ ad potestatem $\frac{1}{n}$, exprimi haec poterit per $A + B\sqrt{-1}$, designantibus A, & B quantitates reales; ac propterea facta substitutione invenietur $x = \frac{v}{2} \times \overline{A+B\sqrt{-1}} + \frac{1}{2v \times \overline{A+B\sqrt{-1}}}$.

Quoniam vero divisa unitate per $v$, seu per radices gradus n unitatis ipsius, quotientes sunt radices eaedem unitatis (Probl. I.), inde sequitur quantitatem $\frac{1}{v} \times \frac{1}{A+B\sqrt{-1}}$ exprimere radices omnes possibiles gradus n quantitatis $\frac{1}{l+\sqrt{1-l^2} \times \sqrt{-1}}$, vel quantitatis aequalis $l - \sqrt{l^2-1}$, cumque sit $\frac{1}{v} \times \frac{1}{A+B\sqrt{-1}} = \frac{1}{v} \times \frac{A-B\sqrt{-1}}{A^2+B^2}$, & simul $\frac{1}{v} \times \overline{A-B\sqrt{-1}}$ adaequet, ut patet, radices omnes gradus n quantitatis

l —



$l - \sqrt{l^2-1}$, facile dignoscitur esse debere, $A^2 + B^2 = 1$, & quantitates reales A, & B singulas unitate minores. Erit itaque $x = \frac{v}{2} \times \overline{A+B\sqrt{-1}} + \frac{1}{2v} \times \overline{A-B\sqrt{-1}}$, dummodo $v$ designet in utroque termino unam, eandemque radicem unitatis. Jam si n est numerus par, radices unitatis gradus n sunt (Probl. I.) $\pm 1$, $a \pm b\sqrt{-1}$, $e \pm f\sqrt{-1}$, $d \pm h\sqrt{-1}$ &c., qui sunt valores ipsius $v$, factisque in aequatione superiori horum valorum substitutionibus pro $v$, erit $x = A$, $x = -A$, $x = Aa - Bb$, $x = Aa + Bb$, $x = Ae - Bf$, $x = Ae + Bf$, $x = Ad - Bh$, $x = Ad + Bh$, $x =$ &c., adeoque valores omnes ipsius x reales erunt, si n sit numerus par. Si n sit numerus impar valores ipsius $v$ erunt (Probl. I.) $+1$, $g \pm k\sqrt{-1}$, $p \pm q\sqrt{-1}$ &c. quibus subrogatis habebitur $x = A$, $x = Ag - Bk$, $x = Ag + Bk$, $x = Ap - Bq$, $x = Ap + Bq$, $x =$ &c. qui rursus omnes reales sunt. Esto nunc l quantitas negativa unitate minor, eritque in hac hypothesi $\overline{x + \sqrt{x^2-1}}^{\,n} = -l + \sqrt{l^2-1} = -1 \times \overline{l - \sqrt{l^2-1}}$, & $x + \sqrt{x^2-1} = u \times \overline{l - \sqrt{l^2-1}}^{\,\frac{1}{n}}$, ubi u designat radices omnes unitatis negati-

va,

væ, ſeu $-1$, unde infertur $x = \frac{u}{2} \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}} + \frac{1}{2u} \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}}$. Porro $\overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}}$ exprimi poteſt per $A - B\sqrt{-1}$, inſtitutaque eadem, ac ſupra, ratiocinatione, invenitur $x = \frac{u}{2} \times \overline{A - B\sqrt{-1}} + \frac{1}{2u} \times \overline{A + B\sqrt{-1}}$, ſimulque $A^2 + B^2 = 1$. Jam valores ipſius u, ſeu radices unitatis negativæ, ubi n eſt par, ſunt (Coroll. III. & IV. Probl. I.) $\mu \pm \nu\sqrt{-1}$, $\lambda \pm \Delta\sqrt{-1}$ &c., ergo ſubrogatione facta erit $x = A\mu + B\nu$, $x = A\mu - B\nu$, $x = A\lambda + B\Delta$, $x = A\lambda - B\Delta$, $x =$ &c. Si vero n ſit impar, valores ipſius u ſunt (Coroll. cit.) $-1$, $\phi \pm \delta\sqrt{-1}$, $\omega \pm \Theta\sqrt{-1}$ &c.; parique ratione ſubſtitutione facta, prodibit $x = -A$, $x = A\phi + B\delta$, $x = A\phi - B\delta$, $x = A\omega + B\Theta$, $x = A\omega - B\Theta$, $x =$ &c. Ergo valores omnes ipſius x, etiam ubi l ſit quantitas negativa unitate minor, reales erunt. Denique ſi $l = 1$, etiam $l + \sqrt{l^2 - 1} = 1$, adeoque $A = 1$, $B = 0$, $x = \frac{u}{2} + \frac{1}{2u}$. Porro, ſi n eſt par, valores u ſunt $\pm 1$, $a \pm b\sqrt{-1}$, $e \pm f\sqrt{-1}$ &c., quibus ſubrogatis fiet $x = 1$,

x



$x = -1$, $x = a$, $x = a$, $x = e$, $x = e$, $x =$ &c. Si n fuerit impar, valores u erunt $+1$, $g \pm k\sqrt{-1}$, $p \pm q\sqrt{-1}$, &c., adeoque $x = 1$, $x = g$, $x = g$, $x = p$, $x = p$, $x =$ &c. Eſto $l = -1$, eritque ut ſupra $A = 1$, $B = 0$, $x = \frac{u}{2} + \frac{1}{2u}$. Jam valores ipſius u, ſeu radices unitatis negativæ gradus n ſunt, ſi n fuerit par, $\mu \pm \nu\sqrt{-1}$, $\lambda \pm \Lambda\sqrt{-1}$ &c. quibus ſubrogatis prodibit $x = \mu$, $x = \mu$, $x = \lambda$, $x = \lambda$, $x =$ &c. Si n fuerit impar, valores u ſunt $-1$, $\phi \pm \delta\sqrt{-1}$, $\omega \pm \Theta\sqrt{-1}$ &c. quibus de more ſubſtitutis fiet $x = -1$, $x = \phi$, $x = \phi$, $x = \omega$, $x = \omega$, $x =$ &c. Itaque valores omnes ipſius x in quocumque caſu, modo l non excedat $\pm 1$, reales erunt, & numero n. Quod ſi l fuerit unitate poſitiva major, bini tantum valores ipſius x reales erunt, ubi n fuerit par; unus, ſi n fuerit impar, & reliqui omnes imaginarii; & ſi l unitate negativa major fuerit, n par, valores omnes x imaginarii erunt; & ſi n fuerit impar, unus tantum realis erit, ceteri imaginarii, ut analyſis veſtigia relegenti innoteſcet. Ajo nunc, valores omnes ipſius x, ubi l non excedat $\pm 1$, non eſſe majores $\pm 1$; quod ſic oſtenditur. Sit primo $l = 1$, n par, valores x ſunt $\pm 1$, $-1$, a, a, e, e, &c.; ſi n impar, ſunt $+1$, g, g, p, p, &c. Sit ſecundo $l = -1$, n par, valores ipſius x erunt, $\mu, \mu, \lambda, \lambda$, &c.; Si n im-

impar, erunt $-1$, $\phi$, $\phi$, $\omega$, $\omega$, &c.; sunt autem a, e, &c., g, p, &c., $\mu$, $\lambda$, &c., $\phi$, $\omega$, &c. unitate minores (Probl. I.). Sit denique l unitate positiva minor, n par, valores x erunt $+A$, $-A$, $Aa - Bb$, $Aa + Bb$, $Ae - Bf$, $Ae + Bf$, &c.. Jam $+A$, & $-A$ sunt unitate minores ex demonstratis, & quia etiam a, b, &c. sunt unitate minores (Probl. I.), erit a fortiori ex fractionum natura $Aa - Bb$ unitate minor, modo $Aa$, & $Bb$ positivæ quantitates sint; si $Aa$, & $Bb$ negativæ quantitates fuerint, tum pro $Aa - Bb$, scribi poterit $-Aa + Bb$, quæ ob eandem rationem unitate minor erit. Quod si $Aa$ sit quantitas positiva, $Bb$ negativa, aut vicissim, $Aa - Bb$ fiet, vel $Aa + Bb$ vel $-Aa - Bb$. Jamvero ex demonstratis $A^2 + B^2 = 1$, & (Probl. I.) $a^2 + b^2 = 1$; ergo $A^2a^2 + B^2b^2 + A^2b^2 + B^2a^2 = 1$; & quoniam $\overline{Ab - Ba}^2$ est semper, ut patet, quantitas positiva, iccirco $A^2b^2 - 2BAba + B^2a^2$, vel erit nihilo æqualis, vel nihilo major, adeoque $A^2b^2 + B^2a^2$, vel $= 2BAba$, vel $> 2BAba$; ex quo consequitur $A^2a^2 + B^2b^2 + 2BAba$ vel esse unitati æqualem, vel unitate minorem, eductaque radice $Aa + Bb$ vel eidem unitati æqualem esse, vel eadem minorem. Eadem facta ratiocinatione, etiam ubi n sit impar, & l quantitas negativa, planum fiet, neminem ex valoribus x unitatem excedere, quando l non excedit $\pm 1$. Ex quo tandem colligitur, valores omnes ipsius x, ubi l non excedit $\pm 1$, reales esse, numero n, nec majores $\pm 1$. Q. E. I.





## PORISMA.

Sit l cosinus arcus dati radio 1 descripti, aliorumque proinde in infinitum; valores omnes ipsius x eruti ex æquatione $l + \sqrt{l^2 - 1} = \overline{x + \sqrt{x^2 - 1}}^{\,n}$ erunt totidem cosinus eorundem arcuum per n divisorum.

### DEM.

Esto BNI (Fig. 1.) circulus radio 1 descriptus, A centrum, $AC = l$, AD æqualis uni ex valoribus x, quos unitatem non excedere (Probl. II.) invenimus, quando l non est unitate major, ut hic contingit. Jam arcus, quorum cosinus est AC, sunt numero infiniti, quia non modo arcus BN eundem cosinum habet, sed etiam BIRN utpote æqualis BNRV, cujus, ut patet, cosinus est eadem AC; sicque eundem cosinum AC habet arcus compositus ex tota peripheria, & arcu BN, ex tota peripheria, & arcu BIRN, ex dupla peripheria, & arcu BN, ex dupla peripheria,

& arcu BIRN, ex tripla peripheria, & arcu BN, ex tripla peripheria, atque ita porro. Si itaque peripheria dicatur P, arcus BN dicatur D, infinita series arcuum habentium cosinum AC erit D, P — D, P + D, 2P — D, 2P + D, 3P — D, 3P + D, &c., arcus autem, quorum cosinus esse debent valores ipsius x, erunt, $\frac{D}{n}$, $\frac{P-D}{n}$, $\frac{P+D}{n}$, $\frac{2P-D}{n}$, $\frac{2P+D}{n}$, $\frac{3P-D}{n}$, $\frac{3P+D}{n}$, &c., qui seriem infinitam constituunt, cujus dignoscitur progressus. Si horum arcuum accipiatur numerus = n, arcus alii post numerum hunc consequentes eundem habebunt cosinum atque arcus proxime antecedentes citra numerum n. Reapse sumpto in serie superiori terminorum numero n, terminus ille, qui locum n in eadem serie occupabit, erit $\frac{\frac{n}{2}P-D}{n}$, si n fuerit par; & $\frac{\overline{\frac{n-1}{2}}\times P+D}{n}$, ubi n fuerit impar, ut seriei progressum consideranti apparebit. Jam arcus, qui proxime consequitur arcum $\frac{\frac{n}{2}P-D}{n}$, est $\frac{\frac{n}{2}P+D}{n}$, eundemque cosinum habet, atque arcus antecedens



dens $\frac{\frac{n}{2}P-D}{n}$; quia si a peripheria P auferatur arcus $\frac{\frac{n}{2}P-D}{n}$ superest arcus $\frac{\frac{n}{2}P+D}{n}$, cujus iccirco cosinus idem est, ac cosinus arcus $\frac{\frac{n}{2}P-D}{n}$; quilibet enim arcus, ut constat, eundem habet cosinum atque arcus complementi ad peripheriam. Eadem ratione arcus, qui proxime consequitur arcum $\frac{\frac{n}{2}P+D}{n}$, est $\frac{\overline{\frac{n}{2}+1}\times P-D}{n}$, qui pariter ob eandem rationem eundem cosinum habebit atque arcus $\frac{\overline{\frac{n}{2}-1}\times P+D}{n}$, qui proxime antecedit arcum $\frac{\frac{n}{2}P-D}{n}$; atque ita porro arcus $\frac{\overline{\frac{n}{2}+1}\times P+D}{n}$, $\frac{\overline{\frac{n}{2}+2}\times P-D}{n}$, $\frac{\overline{\frac{n}{2}+2}\times P+D}{n}$, &c. qui proxime sequuntur post arcum $\frac{\overline{\frac{n}{2}+1}\times P-D}{n}$, eosdem respective cosinus habent atque arcus $\frac{\overline{\frac{n}{2}-1}\times P-D}{n}$, $\frac{\overline{\frac{n}{2}-2}\times P+D}{n}$,

$\frac{\frac{n-1}{2}\times P - D}{n}$, &c. qui comprehenduntur inter $\frac{D}{n}$, & $\frac{\frac{n}{2}P - D}{n}$; ac denique arcus, qui locum n occupat post $\frac{\frac{n}{2}P - D}{n}$, seu terminus 2 n seriei superioris, qui erit $\frac{nP - D}{n}$, eundem cosinum habet atque arcus $\frac{D}{n}$. Jam vero quoniam valores ipsius x ex æquatione $l + \sqrt{l^2 - 1} = \overline{x + \sqrt{x^2 - 1}}^n$ inventi sunt reales, numero n, & inter se diversi, satis modo erit ostendere, horum valorum quemlibet cosinum esse respective arcuum seriei $\frac{D}{n}$, $\frac{P - D}{n}$, $\frac{P + D}{n}$, $\frac{2P - D}{n}$, $\frac{2P + D}{n}$, &c. Ut id calculo finito ostendatur, in æquatione $l + \sqrt{l^2 - 1} = \overline{x + \sqrt{x^2 - 1}}^n$ pro l subrogetur $\frac{1}{\sqrt{u^2 + 1}}$, & $\frac{1}{\sqrt{z^2 + 1}}$ pro x, quo facto prodit

dit



dit $\frac{1 + u\sqrt{-1}}{\sqrt{u^2 + 1}} = \frac{\overline{1 + z\sqrt{-1}}^n}{\overline{\sqrt{z^2 + 1}}^n}$, & $\frac{\overline{1 + u\sqrt{-1}}^2}{u^2 + 1} = \frac{\overline{1 + z\sqrt{-1}}^{2n}}{\overline{1 + z^2}^n}$, seu $\frac{1 + u\sqrt{-1}}{1 - u\sqrt{-1}} = \frac{\overline{1 + z\sqrt{-1}}^n}{\overline{1 - z\sqrt{-1}}^n}$, unde infertur $u\sqrt{-1} = \frac{\overline{1 + z\sqrt{-1}}^n - \overline{1 - z\sqrt{-1}}^n}{\overline{1 - z\sqrt{-1}}^n + \overline{1 + z\sqrt{-1}}^n}$; factoque brevitatis gratia $z\sqrt{-1} = r$ erit

$$u\sqrt{-1} = \frac{\overline{1 + r}^n - \overline{1 - r}^n}{\overline{1 - r}^n + \overline{1 + r}^n} =$$

$$\frac{2nr + 2n\frac{\times\overline{n-1}\times\overline{n-2}}{2\times 3}r^3 + 2n\frac{\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times 3\times 4\times 5}r^5 \text{ \&c.}}{2 + 2n\frac{\times\overline{n-1}}{2}r^2 + 2n\frac{\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times 3\times 4}r^4 + \text{\&c.}}$$

proindeque $u = nz - n\frac{\times\overline{n-1}\times\overline{n-2}}{2\times 3}z^3 + n\frac{\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times 3\times 4\times 5}z^5 - \text{\&c.}$

$$1 - \frac{n\times\overline{n-1}}{2} z^2 + \frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3\times4} z^4 \&c.$$

$$\& \pm u = \frac{\pm nz \mp \frac{n\times\overline{n-1}\times\overline{n-2}}{2\times3} z^3 \pm \frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times3\times4\times5} z^5 \mp \&c.}{1 - \frac{n\times\overline{n-1}}{2} z^2 + \frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3\times4} z^4 \&c.}$$

porro secundum hujus æquationis membrum exprimit tangentem arcus n:pli, cujus simpli tangens sit z, ut Analystis compertum est; adeoque ± z erit tangens arcus prioris, cujus tangens est ± u, divisi per n. Jam arcuum cosinum x habentium tangens est ± z, & arcuum habentium cosinum l tangens est u; cum enim sit $x = \frac{1}{\sqrt{z^2+1}}$, & $l = \frac{1}{\sqrt{u^2+1}}$, erit $z = \frac{\sqrt{1-x^2}}{x}$, $u = \sqrt{\frac{1-l^2}{l}}$ proindeque ±z, & ± u erunt tangentes arcuum respondentium cosinus x, l habentium. Itaque x est cosinus arcus cosinum l habentis divisi per n; ac proinde quilibet ex valoribus ipsius x cosinus est arcus cosinum l habentis divisi per n, sive arcus respondentis seriei inventæ $\frac{D}{n}$, $\frac{P-D}{n}$, $\frac{P+D}{n}$, $\frac{2P-D}{n}$, $\frac{2P+D}{n}$, &c.

Er-



Ergo valores singuli jam inventi ipsius x sunt cosinus singulorum arcuum respective $\frac{D}{n}$, $\frac{P-D}{n}$, $\frac{P+D}{n}$, &c. usque ad $\frac{\frac{n}{2}P - D}{n}$ inclusive, si n est par; vel usque ad $\frac{\frac{\overline{n-1}}{2}\times P + D}{n}$ inclusive, si n est numerus impar. Q. E. D.

Hisce in antecessum constitutis aggrediamur Cotesiani Theorematis demonstrationem.

Esto $l = 1$; tum arcus D æqualis fiet toti peripheriæ, vel nihilo, vel &c., & series arcuum $\frac{P}{n}$, $\frac{P-D}{n}$, $\frac{P+D}{n}$, $\frac{2P-D}{n}$, $\frac{2P+D}{n}$, &c. mutabitur in o, $\frac{P}{n}$, $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$, $\frac{3P}{n}$, $\frac{4P}{n}$, $\frac{4P}{n}$ ....... $\frac{P}{2}$, vel in o, $\frac{2P}{2n}$, $\frac{2P}{2n}$, $\frac{4P}{2n}$, $\frac{4P}{2n}$, $\frac{6P}{2n}$, $\frac{6P}{2n}$, $\frac{8P}{2n}$, $\frac{8P}{2n}$ ...... $\frac{nP}{2n}$, ubi n fuerit par, quorum arcuum cosinus erunt (Probl. II.) +1, −1, a, a, e, e, &c..... Si vero n impar fuerit, arcuum o, $\frac{2P}{2n}$, $\frac{2P}{2n}$, $\frac{4P}{2n}$, $\frac{4P}{2n}$, $\frac{6P}{2n}$, $\frac{6P}{2n}$ ...... $\frac{\overline{n-1}\times P}{2n}$ cosinus erunt (Probl. II.) 1, g, g, p, p, &c.

F 4 Esto

Esto $l = -1$, tum arcus D erit $\frac{p}{2}$, & valores ipsius x, ubi n fuerit par, seu (Probl. II.) $\mu, \mu, \lambda, \lambda$, &c. erunt cosinus arcuum $\frac{p}{2n}, \frac{p}{2n}, \frac{3p}{2n}, \frac{3p}{2n}, \frac{5p}{2n}, \frac{5p}{2n} \ldots\ldots \frac{\overline{n-1} \times p}{2n}$; & supposito n impari, valores ipsius x, seu (Probl. cit.) $-1, \phi, \phi, \omega, \omega$, &c. cosinus erunt arcuum $\frac{p}{2n}, \frac{p}{2n}, \frac{3p}{2n}, \frac{3p}{2n}, \frac{5p}{2n}, \frac{5p}{2n} \ldots\ldots\ldots \frac{np}{2n}$. Describatur (Fig. II., & III.) circulus radio 1, cujus peripheria concipiatur divisa in partes æquales numero 2n, adeoque semiperipheria in partes æquales numero n. Jam, si n sit numerus par, arcus AO, A2, A4, A6, &c., & semiperipheria erunt scilicet arcus o, $\frac{2p}{2n}, \frac{4p}{2n}, \frac{6p}{2n} \ldots\ldots\ldots \frac{np}{2n}$, proindeque arcus AO, A2, A4, A6, &c., & semiperipheria habebunt cosinus $+1, -1, a, e$, &c.; & quia $+1$ est cosinus arcus AO, & $-1$ cosinus semiperipheriæ, iccirco a, e, &c. cosinus erunt reliquorum arcuum A2, A4, A6, &c., & accipi poterit a pro cosinu arcus A2, e pro cosinu arcus A4, &c.: nihil enim detrimenti patitur demonstratio, quamvis a, e, &c. cosinus sint diversorum arcuum superioris seriei: ar-



arcus porro A1, A3, A5, &c., & semiperipheria dempto arcu $\frac{p}{2n}$ habebunt cosinus $\mu, \lambda$, &c. Sit modo n numerus impar, tum (Fig. III.) arcus AO, A2, A4, A6, &c., & semiperipheria dempto arcu $\frac{p}{2n}$ habebunt cosinus 1, g, p, &c., cumque 1 sit cosinus arcus AO, erunt g, p, &c. cosinus reliquorum A2, A4, A6, &c.; & arcus A1, A3, A5, &c., & semiperipheria habebunt cosinus $-1, \phi, \omega$, &c.; cumque ipsius peripheriæ cosinus sit $-1$, reliquorum iccirco $A^1, A^3, A^5$, &c. cosinus erunt $\phi, \omega$, &c. Jam in circulo (Fig. IV.) radio 1 descripto, sumpto supra diametrum OD puncto C, quod distet a centro A per rectam $AC = x$, ductaque inde ad extremum arcus OM, cujus cosinus sit $BA = a$, recta CM, hæc erit $= \sqrt{1 - 2ax + x^2}$: est enim $BC = a - x$, $BM = \sqrt{1 - a^2}$, & $CM = \sqrt{BC^2 + BM^2} = \sqrt{1 - 2ax + x^2}$. Idem continget, licet arcus ON habeat cosinum negativum $AS = -e$; erit enim $CS = x - e$, adeoque $CN = \sqrt{1 - 2ex + x^2}$: Præterea ipsa quoque CR ducta ad extremum arcus $OR = OM$ erit $= \sqrt{1 - 2ax + x^2}$, & CQ ducta ad ex-

extremum arcus $OQ = ON$ erit $= \sqrt{\overline{1 - 2ex + x^2}}$.

Itaque si accipiatur super diametro A10 (Fig. II.) punctum Q ejusmodi, ut sit $QE = x$, recta Q0, Q2, Q4, Q6..... Q10 ductæ ad extremitates arcuum A0, A2, A4, A6.... A10, quorum cosinus sunt, supposito n pari, 1, a, e..... —1, erunt respective $= 1 - x$, $\sqrt{\overline{1 - 2ax + x^2}}$, $\sqrt{\overline{1 - 2ex + x^2}}$ ......, $1 + x$, rectæque Q18, Q16, Q14, &c., ductæ ad extremitates arcuum A18, A16, &c. respective æqualium arcubus A2, A4, &c., erunt rursus $= \sqrt{\overline{1 - 2ax + x^2}}$, $\sqrt{\overline{1 - 2ex + x^2}}$, &c. Itaque si dividatur peripheria circuli radio 1 descripti in partes æquales numero 2n, & a puncto Q ducantur rectæ ad extremitates illorum arcuum, qui numerum parem earum partium comprehendant, & insuper recta una ad punctum A, a quo incipit divisio, harum rectarum numerus erit n, earumque productum, si n fuerit par, erit

$$= \overline{1 - x} \times \sqrt{\overline{1 - 2ax + x}}^2 \times \sqrt{\overline{1 - 2ax + x}}^2 \times \sqrt{\overline{1 - 2ex + x}}^2$$

$$\times \sqrt{\overline{1 - 2ex + x}}^2 \times \ldots\ldots \times \overline{1 + x} = \overline{1 - x} \times \overline{1 + x}$$

$$\times \overline{1 - 2ax + x}^2 \times \overline{1 - 2ex + x}^2 \times \text{\&c.} = (\text{Coroll. II.}$$

Pro-



Probl. I.) $1 - x^n$; proindeque $1 - x^n$ æquatur producto omnium rectarum, quæ ex puncto Q ducuntur ad extremitates arcuum A0, A2, A4, A6, A8, A10, A12 ......$A\overline{2n-2}$. Eadem ratione in hypothesi n paris ductis rectis ad extremitates arcuum A1, A3, A5, &c., qui partium illarum, in quas integra peripheria æqualiter divisa fuit, numerum imparem comprehendunt, & quorum cosinus sunt ex demonstratis $\mu$, $\lambda$, &c. rectæ eædem Q1, Q3, &c. erunt $= \sqrt{\overline{1 - 2\mu x + x^2}}$, $\sqrt{\overline{1 - 2\lambda x + x^2}}$, &c., ac totidem erunt rectæ aliæ Q19, Q17, &c. prioribus respective æquales, ductæque ad extremitates aliorum arcuum prioribus æqualium. Igitur Q1 × Q3........

$$\times Q19 \times Q17 \times \text{\&c.} = \overline{1 - 2\mu x + x}^2 \times \overline{1 - 2\lambda x + x^2}^2 \times$$

&c. $=$ (Coroll. IV. Probl. II.) $1 + x^n$; adeoque tandem productum omnium rectarum, quæ ex puncto Q ducuntur ad extremitates omnium arcuum æqualium, & numero 2n, in quos integra peripheria dividitur, computata insuper recta, quæ ducitur ad divisionis initium A, videlicat Q0 × Q1 × Q2

$$\times Q3 \times Q4 \times Q5 \times \text{\&c.} = \overline{1 - x}^n \times \overline{1 + x}^n =$$

$1 - x^{2n}$. Haud absimili ratione in hypothesi n imparis (Fig. III.) demonstratur productum re-

rectarum, quæ ex puncto Q ducuntur ad extremitates arcuum A0, A2, A4, A6 &c. comprehendentium numerum parem illarum partium, quarum peripheria continet 2n, computata etiam recta ad divisionis initium ducta, æquale esse $1-x^n$; & productum aliarum, quæ ad reliquorum arcuum extremitates ducuntur, æquari $1+x^n$; ac demum productum rectarum omnium, quæ ad arcuum omnium æqualium, in quos peripheria divisa fuit, exstremitates ducuntur, æquale esse $1-x^{2n}$.

Persequamur modo postremam Cotesiani Theorematis partem, quæ pleno alveo ex demonstratis fluit. Ostendendum igitur est, trinomium $x^{2n} \pm 2lx^n + 1$ resolvi semper posse in factores reales secundi gradus, quorum terminus secundus coefficientem habeat pendentem a divisione circuli in arcus æquales. Supponatur $x^{2n} \pm 2lx^n + 1 = 0$; unde eruitur $x^n = \mp l \pm \sqrt{\overline{l^2-1}}$; ex quo patet $x^n$ esse quantitatem realem, ubi l excedit unitatem, imaginariam, si l unitate minor sit. Fiat l unitate major, eritque $x^{2n} + 2lx^n + 1 = \overline{x^n + l - \sqrt{\overline{l^2-1}}} \times \overline{x^n + l + \sqrt{\overline{l^2-1}}}$ = facto ex duobus binomiis realibus. Jam si quantitas positiva realis $l - \sqrt{\overline{l^2-1}}$, fiat $= p^n$, & $l + \sqrt{\overline{l^2-1}} = q^n$, $\frac{x}{p} = z$, $\frac{x}{q} = y$, prodibit $x^{2n} + 2lx^n + 1 = p^n \times \overline{z^n+1} \times q^n \times \overline{y^n+1}$; ac tum $z^n + 1$, tum $y^n + 1$ in factores reales secundi gradus (Coroll. IV. Probl. I.) resolvuntur; ergo & trinomium ipsum $x^{2n} + 2lx^n + 1$ in eosdem resolvetur. Eadem instituatur operatio in trinomio $x^{2n} - 2lx^n + 1$, quod rursus in factores reales resolubile esse constabit. Si vero l sit unitate minor, tum $x^n$ prodibit $= -l \pm \sqrt{l^2-1}$, seu quantitati imaginariæ, si trinomium fuerit $x^{2n} + 2lx^n + 1 = 0$, eductaque utrimque radice gradus n, fiet $x = u \times \overline{l - \sqrt{l^2-1}}^{\frac{1}{n}}$ pariterque $x = u \times \overline{l + \sqrt{l^2-1}}^{\frac{1}{n}}$, designante u radices omnes gradus n unitatis negativæ, seu $-1$, quas numero n æquales supra demonstravimus. Hinc divisores simplices trinomii $x^{2n} + 2lx^n + 1$ erunt numero 2n, seu $x - u \times \overline{l - \sqrt{l^2-1}}^{\frac{1}{n}}$, & $x - u \times \overline{l + \sqrt{l^2-1}}^{\frac{1}{n}}$, quorum uterque numerum n divisorum complectitur. Jamvero quoniam valores ipsius u tales sunt (Probl. I.), ut, ut, si per istorum quemlibet unitatem dividas, prodeat in quotiente alter ex valoribus ejusdem u, atque ita, divisa unitate per omnes successive valores ipsius u, valores iidem omnes mutato licet ordine oriantur; idcirco divisorum simplicium trinomii $x^{2n} \ast 2lx^{n} \ast 1$ numerus n exprimetur per

$x - u \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}}$; ac divisores alii numero pariter n, seu $x - u \times \overline{l + \sqrt{l^2 - 1}}^{\frac{1}{n}}$ exprimi poterunt per $x - \frac{1}{u} \times \overline{l + \sqrt{l^2 - 1}}^{\frac{1}{n}}$; quia si in expressione $x - \frac{1}{u} \times \overline{l + \sqrt{l^2 - 1}}$ omnes successive valores ipsius u substituantur, eædem quantitates prodibunt, quæ iisdem factis substitutionibus in formula altera $x - u \times \overline{l + \sqrt{l^2 - 1}}^{\frac{1}{n}}$ oriuntur. Atque inde porro deducitur, valores omnes $x - u \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}}$ & $x - \frac{1}{u} \times \overline{l + \sqrt{l^2 - 1}}^{\frac{1}{n}}$ in se invicem ductos efficere $x^{2n} \ast 2lx^{n} \ast 1$. Accipiantur

nunc



nunc valores singuli formulæ $x - u \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}}$, & horum quilibet ducatur in formulæ alterius $x - \frac{1}{u} \times \overline{l + \sqrt{l^2 - 1}}^{\frac{1}{n}}$ valorem hujusmodi, ut u idem valeat utrobique; obtinebitur numerus n divisorum secundi gradus, qui omnes exprimentur per formulam $x^2 - 2x$

$$\frac{\left(u \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}} + \frac{1}{u} \times \overline{l + \sqrt{l^2 - 1}}^{\frac{1}{n}}\right)}{2} + 1,$$

in qua subrogatis singulis valoribus ipsius u, prodibunt factores omnes secundi gradus dati trinomii, quorum productum eidem trinomio $x^{2n} + 2lx^{n} + 1$ æquale erit. Porro

$$\frac{u \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}} \ast \frac{1}{u} \times \overline{l + \sqrt{l^2 - 1}}^{\frac{1}{n}}}{2} = \frac{u}{2} \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}} + \frac{1}{2u \times \overline{l - \sqrt{l^2 - 1}}^{\frac{1}{n}}},$$

quæ (Poris.) exprimit cosinus omnes numero n arcuum seriei $\frac{D}{n}$, $\frac{P - D}{n}$, $\frac{P + D}{n}$, $\frac{2P - D}{n}$, $\frac{2P + D}{n}$, &c.

usque

usque ad arcum $\frac{\frac{n}{2}P - D}{2}$, si n fuerit par, & usque ad arcum $\frac{\frac{n-1}{2} \times P + D}{n}$, ubi n fuerit impar, designante P peripheriam circuli radio 1 descripti, D arcum, cujus cosinus sit — l. Si ergo horum arcuum cosinus vocentur $\psi$, $\pi$, $\rho$, &c. valores quantitatis

$$x^2 - 2x\frac{\left(u\times\overline{l-\sqrt{l^2-1}}^{\frac{1}{n}} + \frac{1}{u}\times\overline{l+\sqrt{l^2-1}}^{\frac{1}{n}}\right)}{2} + 1$$

orti ex substitutione valorum omnium ipsius u erunt $x^2-2\psi x+1$, $x^2-2\pi x+1$, $x^2-2\rho x+1$, &c. numero n, quorum proinde productum restituet trinomium $x^{2n}+2lx^n+1$. Itaque trinomium $x^{2n}+2lx^n+1$ resolvi semper poterit in divisores reales secundi gradus, supposita arcuum circularium divisione. Eadem omnino ratione trimonii $x^{2n}-2lx^n+1$ invenientur divisores simplices $x - v\times\overline{l+\sqrt{l^2-1}}^{\frac{1}{n}}$, $x - v\times\overline{l-\sqrt{l^2-1}}^{\frac{1}{n}}$, designante v radices omnes gradus n unitatis positivæ, quorum alter $x - v\times\overline{l-\sqrt{l^2-1}}^{\frac{1}{n}}$





ob rationem jam indicatam exprimi etiam poterit per $x - \frac{1}{v}\times\overline{l-\sqrt{l^2-1}}^{\frac{1}{n}}$, & factum ex ipsis, nimirum $x^2 - 2x\frac{\left(v\times\overline{l+\sqrt{l^2-1}}^{\frac{1}{n}} + \frac{1}{v}\times\overline{l-\sqrt{l^2-1}}^{\frac{1}{n}}\right)}{2} + 1$ exprimet factores omnes secundi gradus dati trinomii $x^{2n}-2lx^n+1$. Porro $\frac{\left(v\times\overline{l+\sqrt{l^2-1}}^{\frac{1}{n}} + \frac{1}{v}\times\overline{l-\sqrt{l^2-1}}^{\frac{1}{n}}\right)}{2}$ exprimit (Poris.) cosinus omnes numero n arcuum $\frac{D}{n}$, $\frac{P-D}{n}$, $\frac{P+D}{n}$, &c., sumpto arcu D, cujus cosinus sit + l. Propterea si arcuum istorum cosinus fuerint m, n, r, &c. habebitur $\overline{x^2-2mx+1}\times\overline{x^2-2nx+1}\times\overline{x^2-2rx+1}\times$ &c. $= x^{2n}-2lx^n+1$; proindeque trinomium $x^{2n}-2lx^n+1$ resolubile erit in factores secundi gradus numero n, supposita circuli divisione in partes æquales. Quod erat ultimo loco demonstrandum.

Tempus nunc est amplissimum Cotesiani

Theorematis fæcunditatem in universa Analysi Sublimiori ob oculos ponere, ejusque latissimos usus aperire; quod in sequentibus Corollariis præstare conabimur.

### COROLLARIUM I.

Magnus Eulerus (Eulerum autem cum nomino, apicem quendam humanæ subtilitatis, quod de Archimede dictum est, totiusque Mathematicæ Disciplinæ absolutionem animo concipio) in Opusculo nunquam satis laudando *De la Controverse entre Mrs. Leibnitz, & Bernoulli sur les Logarithmes des nombres negatif & imaginaires, Memoir. de l'Acad. Roy. des Scienc. & Bell. Lettr. de Berlin année* 1749 veluti notum, & demonstratum ponit, formulæ $p^n - q^n$, in qua n numerum quemlibet integrum designat, factorem quemcumque exprimi per $p^2 - 2pq\cos.\frac{2\lambda\pi}{n} + q^2$, ubi $\lambda$ numerum quemvis integrum, atque etiam nihilum indigitat, $\pi$ vero angulum 180°, seu dimidiam circuli peripheriam, cujus radius sit $= 1$. Hoc ex superius demonstratis tam liquido fluit, ut temporis jacturam facerem, si nova demonstratione confirmarem. Ex hoc vero deducit Eulerus $p =$



$p = q\left(\cos.\frac{2\lambda\pi}{n} \pm \sin.\frac{2\lambda\pi}{n}\sqrt{-1}\right)$, quia facta $p^2 - 2pq\cos.\frac{2\lambda\pi}{n} + q^2 = 0$, & resoluta de more æquatione; obtinetur $p = q\cos.\frac{2\lambda\pi}{n} \pm q\sqrt{\left(\cos.\frac{2\lambda\pi}{n}\right)^2 - 1}$, seu (quia $1 - \left(\cos.\frac{2\lambda\pi}{n}\right)^2 = \left(\sin.\frac{2\lambda\pi}{n}\right)^2$, adeoque $\left(\cos.\frac{2\lambda\pi}{n}\right)^2 - 1 = -\left(\sin.\frac{2\lambda\pi}{n}\right)^2$) $p = q\left(\cos.\frac{2\lambda\pi}{n} \pm \sin.\frac{2\lambda\pi}{n}\sqrt{-1}\right)$.

### COROLL. II.

Laudatus Geometra in citato Opusculo ait, radices omnes alterius formulæ $p^n + q^n$ obtineri per resolutionem formulæ $p^2 - 2pq\cos\frac{(2\lambda - 1)\pi}{n} + q^2$, easdem ac supra quantitates designantibus $\lambda$, & $\pi$ nisi quod $\lambda$ in hoc casu nunquam fit $= 0$. Hoc pariter adeo evidens est, & perspicuum ex jam demonstratis, ut nova demonstratione non egeat. Hinc autem, ut supra, colligitur $p = q\left(\cos.\frac{(2\lambda - 1)\pi}{n} \pm \sin.\frac{(2\lambda - 1)\pi}{n}\sqrt{-1}\right)$, unde

unde eruuntur radices omnes formulæ $p^n + q^n$, subrogatis successive loco ipsius $\lambda$ numeris integris respondentibus. Divisores autem simplices formulæ ejusdem erunt $p - q\left(\cos.\frac{(2\lambda-1)\pi}{n} \pm \text{sin}.\frac{(2\lambda-1)\pi}{n}\sqrt{-1}\right)$

COROLL. III.

Sit Formula $a^n - z^n$

$n = 2$

Formulæ

$a^2 - z^2$

Factores erunt

$a - z$
$a + z$

Si $n = 4$

Formulæ

$a^4 - z^4$

Factores erunt

$a - z$
$a + z$
$a^2 - 2az\cos.\frac{2}{4}\pi + z^2$

Si $n = 6$

Formulæ

$a^6 - z^6$

Factores erunt

$a - z$
$a + z$
$a^2 - 2az\cos.\frac{2}{6}\pi + z^2$
$a^2 - 2az\cos.\frac{4}{6}\pi + z^2$

Si $n = 8$

Formulæ

$a^8 - z^8$

Factores erunt

$a - z$
$a + z$
$a^2 - 2az\cos.\frac{2}{8}\pi + z^2$
$a^2 - 2az\cos.\frac{4}{8}\pi + z^2$
$a^2 - 2az\cos.\frac{6}{8}\pi + z^2$

Si



Si $n = 3$

Formulæ

$a^3 - z^3$

Factores erunt

$a - z$
$a^2 - 2az\cos.\frac{2}{3}\pi + z^2$

Si $n = 5$

Formulæ

$a^5 - z^5$

Factores erunt

$a - z$
$a^2 - 2az\cos\frac{2}{5}\pi + z^2$
$a^2 - 2az\cos.\frac{4}{5}\pi + z^2$

Si $n = 7$

Formulæ

$a^7 - z^7$

Factores erunt

$a - z$
$a^2 - 2az\cos.\frac{2}{7}\pi + z^2$
$a^2 - 2az\cos.\frac{4}{7}\pi + z^2$
$a^2 - 2az\cos.\frac{6}{7}\pi + z^2$

Si $n = 9$

Formulæ

$a^9 - z^9$

Factores erunt

$a - z$
$a^2 - 2az\cos.\frac{2}{9}\pi + z^2$
$a^2 - 2az\cos\frac{4}{9}\pi + z^2$
$a^2 - 2az\cos.\frac{6}{9}\pi + z^2$
$a^2 - 2az\cos.\frac{8}{9}\pi + z^2$

COROLL. IV.

Sit Formula $a^n + z^n$

$n = 4$

Formulæ

$a^4 + z^4$

Factores erunt

$a^2 - 2az\cos.\frac{1}{4}\pi + z^2$
$a^2 - 2az\cos.\frac{3}{4}\pi + z^2$

Si $n = 3$

Formulæ

$a^3 + z^3$

Factores erunt

$a + z$
$a^2 - 2az\cos.\frac{1}{3}\pi + z^2$



Si

Si $n = 6$

Formulæ

$a^6 + z^6$

Factores erunt

$a^2 - 2az\cos.\frac{1}{6}\pi + z^2$

$a^2 - 2az\cos.\frac{3}{6}\pi + z^2$

$a^2 - 2az\cos.\frac{5}{6}\pi + z^2$

Si $n = 5$

Formulæ

$a^5 \maltese z^5$

Factores erunt

$a \maltese z$

$a^2 - 2az\cos.\frac{1}{5}\pi \maltese z^2$

$a^2 - 2az\cos.\frac{3}{5}\pi \maltese z^2$

Si $n = 8$

Formulæ

$a^8 \maltese z^8$

Factores erunt

$a^2 - 2az\cos.\frac{1}{8}\pi + z^2$

$a^2 - 2az\cos.\frac{3}{8}\pi + z^2$

$a^2 - 2az\cos.\frac{5}{8}\pi + z^2$

$a^2 - 2az\cos.\frac{7}{8}\pi + z^2$

Si $n = 7$

Formulæ

$a^7 + z^7$

Factores erunt

$a \maltese z$

$a^2 - 2az\cos.\frac{1}{7}\pi \maltese z^2$

$a^2 - 2az\cos.\frac{3}{7}\pi \maltese z^2$

$a^2 - 2az\cos.\frac{5}{7}\pi \maltese z^2$

Si



Si $n = 10$

Formulæ

$a^{10} \maltese z^{10}$

Factores erunt

$a^2 - 2az\cos\frac{1}{10}\pi + z^2$

$a^2 - 2az\cos.\frac{3}{10}\pi + z^2$

$a^2 - 2az\cos.\frac{5}{10}\pi \maltese z^2$

$a^2 - 2az\cos.\frac{7}{10}\pi \maltese z^2$

$a^2 - 2az\cos.\frac{9}{10}\pi \maltese z^2$

Si $n = 9$

Formulæ

$a^9 \maltese z^9$

Factores erunt

$a + z$

$a^2 - 2az\cos.\frac{1}{9}\pi \maltese z^2$

$a^2 - 2az\cos.\frac{3}{9}\pi \maltese z^2$

$a^2 - 2az\cos.\frac{5}{9}\pi + z^2$

$a^2 - 2az\cos.\frac{7}{9}\pi + z^2$

## COROLL. V.

Reſolvamus nunc Hermanniana methodo fractionem $\frac{1}{1+z^n}$ in ſimpliciores, cum exponens n eſt numerus par. Fiat binomium $1 + z^n = u$, ſintque $Q^2$, $R^2$, $S^2$, &c. factores ejus ſupra inventi. Facta differentiatione logarithmica erit $\frac{nz^{n-1}dz}{1+z^n} = \frac{du}{u}$, & $\frac{nz^{n-1}dz}{1+z^n} - \frac{ndz}{z} = \frac{du}{u} - \frac{ndz}{z}$, ſeu $-\frac{ndz}{z\times\overline{1+z^n}} =$

 du

$\frac{dU}{U} = \frac{ndz}{z}$; adeoque $\frac{n}{1+z^n} = -\frac{zdU}{Udz} + n$. Jamvero $1+z^n = Q^2.R^2.S^2.\&c.$, atque iccirco $dU = 2QdQ.R^2.S^2. + 2RdR\,Q^2.S^2. + 2SdS.R^2.Q^2. + \&c.$, & $\frac{dU}{U} = \frac{2dQ}{Q} + \frac{2dR}{R} + \frac{2dS}{S} + \&c.$, & $-\frac{zdU}{Udz} = -\frac{2zdQ}{Qdz} - \frac{2zdR}{Rdz} - \frac{2zdS}{Sdz} - \&c.$; præterea $Q^2 = 1-2\mu z+z^2$, $R^2 = 1-2\lambda z+z^2$, $S^2 = 1-2\varepsilon z+z^2$, &c.; igitur $-\frac{2zdQ}{Qdz} = \frac{2\mu z-2z^2}{1-2\mu z+z^2} = -2+\frac{2-2\mu z}{1-2\mu z+z^2}$; $-\frac{2zdR}{Rdz} = \frac{2\lambda z-2z^2}{1-2\lambda z+z^2} = -2+\frac{2-2\lambda z}{1-2\lambda z+z^2}$; $-\frac{2zdS}{Sdz} = \frac{2\varepsilon z-2z^2}{1-2\varepsilon z+z^2} = -2+\frac{2-2\varepsilon z}{1-2\varepsilon z+z^2}$; &c.; proindeque $-\frac{zdU}{Udz}+n = -2-2-2-\&c. + n + \frac{2-2\mu z}{1-2\mu z+z^2} + \frac{2-2\lambda z}{1-2\lambda z+z^2} + \frac{2-2\varepsilon z}{1-2\varepsilon z+z^2} + \&c.$ Est autem, ut constat, $n-2-2-2\ \&c. = 0$; ergo $-\frac{zdU}{Udz}+n =$

2 —



$\frac{2-2\mu z}{1-2\mu z+z^2} + \frac{2-2\lambda z}{1-2\lambda z+z^2} + \frac{2-2\varepsilon z}{1-2\varepsilon z+z^2} + \&c.$, & $-\frac{zdU}{nUdz}+1 = \frac{1}{1+z^n} = \frac{\frac{2}{n}-\frac{2\mu}{n}z}{1-2\mu z+z^2} + \frac{\frac{2}{n}-\frac{2\lambda}{n}z}{1-2\lambda z+z^2} + \frac{\frac{2}{n}-\frac{2\varepsilon}{n}z}{1-2\varepsilon z+z^2} + \&c.$

Si exponens n impar fuerit, factores binomii $1+z^n$ erunt $1-2\phi z+z^2$, $1-2\omega z+z^2$, $1-2\psi z+z^2$, &c., $1+z$; & factis iisdem ut supra, invenietur $\frac{1}{1+z^n} =$

$$\frac{\frac{2}{n}-\frac{2\phi}{n}z}{1-2\phi z+z^2} + \frac{\frac{2}{n}-\frac{2\omega}{n}z}{1-2\omega z+z^2} + \frac{\frac{2}{n}-\frac{2\psi}{n}z}{1-2\psi z+z^2}\ldots\ldots + \frac{\frac{1}{n}}{1+z}$$

## COROLL. VI.

Haud absimili methodo fractionem $\frac{1}{1-z^n}$ in suas primitivas resolvemus. Sit enim primo n numerus par, factores binomii $1-z^n$ erunt $1-z$, $1+z$, $1-2az+z^2$, $1-2ez+z^2$, $1-2cz+z^2$, &c., adeoque iisdem ac supra

pra

pra institutis, invenietur $\frac{1}{1-z^n} = \frac{\frac{1}{n}}{1-z} + \frac{\frac{1}{n}}{1+z} +$

$$\frac{\frac{2}{n} - \frac{2a}{n}z}{1-2az+z^2} + \frac{\frac{2}{n} - \frac{2e}{n}z}{1-2ez+z^2} +$$

$$\frac{\frac{2}{n} - \frac{2c}{n}z}{1-2cz+z^2} + \text{\&c.}$$

Sit secundo n numerus impar; factores binomii $1-z^n$ erunt $1-z$, $1-2gz+z^2$, $1-2pz+z^2$, $1-2rz+z^2$, &c., proindeque $\frac{1}{1-z^n} = \frac{\frac{1}{n}}{1-z} + \frac{\frac{2}{n} - \frac{2g}{n}z}{1-2gz+z^2} +$

$$\frac{\frac{2}{n} - \frac{2p}{n}z}{1-2pz+z^2} + \frac{\frac{2}{n} - \frac{2r}{n}z}{1-2rz+z^2} + \text{\&c.}$$

## COROLL. VII.

Hinc facile patet modus integrandi fractionem $\frac{dz}{1\pm z^n}$, quæ nullum ex dictis incommodum parit. Integranda sit insuper fractio $\frac{z^m dz}{1\pm z^n}$, in qua index m sit numerus integer affirmativus; tres occurrent casus; vel erit $m = n$, vel $m > n$, vel $m < n$. Si $m = n$, prodibit $\frac{z^m dz}{1\pm z^n} = \pm dz \mp \frac{dz}{1\pm z^n}$, quæ ex dictis integratur. Si vero m sit $>$, vel $<$ n, fractio $\frac{z^m dz}{1\pm z^n}$ (Coroll v. & VI.) in simpliciores resolvetur, & earum quælibet ducetur in $z^m$, & singulæ per notas regulas ad integrationem revocabuntur. Denique si $m = n-1$ ut formula $\frac{z^m dz}{1\pm z^n}$ fiat $\frac{z^{n-1}dz}{1\pm z^n}$, ipsius integrale erit, ut constat, $\frac{1}{n}L\overline{1\pm z^n}$, omissa constanti, quam hic non considero.

Si fractio fuerit $\frac{dz}{z^m \times \overline{1\pm z^n}}$, facta $z = \frac{1}{y}$, mutabitur in $-\frac{y^{m+n-2}dy}{y^n \pm 1}$ quæ ad superiorem revocatur.

## COROLL. VIII.

Si in fractione $\frac{z^m dz}{1 \pm z^n}$ index n negativus extiterit, seu $= -r$, mutabitur fractio in hanc $\frac{z^{m+r} dz}{z^r \pm 1} = z^m dz \mp \frac{z^m dz}{z^r \pm 1}$, quæ integratur ex dictis. Si fractio integranda fuerit $\frac{z^{\frac{r}{t}} dz}{1 \pm z^n}$, fiat $z = y^t$, eritque $\frac{z^{\frac{r}{t}} dz}{1 \pm z^n} = \frac{t y^{t+r-1} dy}{1 \pm y^{nt}}$. Si fuerit $\frac{z^m dz}{1 \pm z^{\frac{t}{r}}}$, fiat rursus $z = y^r$, & prodibit $\frac{z^m dz}{1 \pm z^{\frac{t}{r}}} = \frac{r y^{mr+r-1} dy}{1 \pm y^t} = \pm r y^{mr+r-t-1} dy \mp \frac{r y^{mr+r-t-1} dy}{1 \pm y^t}$, quæ ex præmissis integratur.

Sit denique $\frac{z^{\frac{r}{p}} dz}{1 \pm z^{\frac{t}{q}}}$; fiat $z = y^{pq}$, & fractio proposita mutabitur in $\frac{pq y^{qr+pq-1} dy}{1 \pm y^{pt}} = \pm pq y^{qr+pq-pt-1} dy \mp \frac{pq y^{qr+pq-pt-1} dy}{1 \pm y^{pt}}$, quæ consuetam induit formam.



## COROLL. IX

Integranda sit Formula $\frac{z^n dz}{z^m \pm a^{mr}}$; fiat $S\frac{z^n dz}{z^m \pm a^{mr}} = B z^{n+rm-2m+1} + \frac{C z^{n+rm-2m}}{z^m \pm a^m} + \frac{D z^{n+rm-2m-1}}{(z^m \pm a^m)^{r-1}} \dots + K + H S\frac{z^n dz}{z^m \pm a^m}$. Differentiata hac æquatione, & nihilo æquata, factisque singulis terminis nihilo æqualibus, eruentur valores assumptarum B, C, D.... K, H; proindeque integratio Formulæ $\frac{z^n dz}{z^m \pm a^{mr}}$ pendebit semper ab integratione formulæ $\frac{H z^n dz}{z^m \pm a^m}$, quam supra integrare docuimus.

Occurret quandoque, ut assumptarum aliqua B, C, D, &c. arbitraria inveniatur, quan-

quando scilicet $n > m - 1$. Praeterea Formula proposita algebraice integrabitur ubi $m = n + 1$, evanescente tunc assumpta H. Praestabit rem exemplis illustrare ex Curvarum rectificatione, & quadratura depromptis.

## EXEMPLUM I.

Rectificanda proponatur Parabola Apolloniana. Constat, arcum Parabolicum, cujus abscissa est x, aequalem esse $\int \frac{dx\sqrt{4x^2+ax}}{2x}$.

Fiat de more $\sqrt{4x^2+ax} = xz$, eritque

$$\int \frac{dx\sqrt{4x^2+ax}}{2x} = \int \frac{-az^2dz}{(z^2-4)^2}.$$

Fiat jam ex Canone superius tradito $\int\frac{z^2dz}{(z^2-4)^2} = \frac{Bz^3+Cz^2+Dz+K}{z^2-4} + H\int\frac{z^2dz}{z^2-4}$. Instituta differentiatione, prodibit

$$\begin{matrix} Bz^4dz & -Dz^2dz \\ +Hz^4dz & -12Bz^2dz \\ & -4Hz^2dz \\ & -z^2dz \end{matrix}$$





$\begin{matrix} +8Czdz \\ -2Kzdz \end{matrix} - 4Dz = 0$, & singulis terminis nihilo aequatis, invenietur $H = \frac{1}{8}$, $B = -\frac{1}{8}$, $C = -\frac{K}{4}$, $D = 0$, K vero erit arbitraria, quia hic occurrit $n > m - 1$, seu $2 > 2 - 1$. Erit itaque

$$\int\frac{z^2dz}{(z^2-4)^2} = -\frac{\frac18 z^3 - \frac K4 z^2 + K}{z^2-4} + \frac18\int\frac{z^2dz}{z^2-4},$$

adeoque

$$\int\frac{-az^2dz}{(z^2-4)^2} = \frac{\frac18 az^3 + \frac{Ka}{4}z^2 - aK}{z^2-4} - \frac a8\int\frac{z^2dz}{z^2-4} = \frac{\frac18 az^3}{z^2-4} + \frac{aK}{4} - \frac a8\int\frac{z^2dz}{z^2-4};$$

est autem

$$-\frac a8\int\frac{z^2dz}{z^2-4} = \int -\frac{adz}{8} - \frac a2\int\frac{dz}{z^2-4} = -\frac{az}{8} + \frac18 a\,L\frac{z+2}{z-2} = -\frac18\,\frac{az^3 + \frac12 az}{z^2-4} + \frac18 a\,L\frac{z+2}{z-2};$$

igitur

$$\int\frac{-az^2dz}{(z^2-4)^2} = \frac{\frac12 az}{z^2-4} + \frac18 a\,L\frac{z+2}{z-2} +$$

$\frac{aK}{4} + \phi$, ($\phi$ est quantitas constans addenda), subrogatoque loco ipsius z valore $\sqrt{\frac{x^2+ax}{x}}$, prodibit $S\frac{-az^2dz}{\overline{z^2-4}^2}$, sive arcus Parabolicus quaesitus $= \frac{1}{2}\sqrt{4x^2+ax} \pm \frac{a}{8}L\left(\frac{\sqrt{4x^2+ax}+2x}{\sqrt{4x^2+ax}-2x}\right) \pm \frac{aK}{4} = \frac{1}{2}\sqrt{4x^2+ax} + \frac{a}{8}L\left(\frac{\sqrt{4x+a}+2\sqrt{x}}{\sqrt{4x+a}-2\sqrt{x}}\right) \pm \frac{aK}{4} \pm \phi$

Ut jam inveniatur constans $\phi$, supponatur $x = 0$, tumque arcus Parabolicus evanescet, ut constat; eritque $\phi = -\frac{a}{8}L1 - \frac{aK}{4}$; ac denique arcus Parabolicus $= \frac{1}{2}\sqrt{4x^2+ax} + \frac{a}{8}L\left(\frac{\sqrt{4x+a}+2\sqrt{x}}{\sqrt{4x+a}-2\sqrt{x}}\right) = \frac{1}{2}\sqrt{4x^2+ax} \pm \frac{a}{8}L\left(\frac{8x+a+4\sqrt{4x^2+ax}}{a}\right)$.

EXEMPLUM II.

Invenienda proponatur Spatii Cissoidalis quadratura. Sit x abscissa Cissoidis, a diameter Circuli genitoris. Notum est, Cissoidis spatium aequale esse $S\frac{x^{\frac{3}{2}}dx}{\sqrt{a-x}} = S\frac{x^2dx}{\sqrt{ax-x^2}}$.

Fiat



Fiat juxta consuetum $\sqrt{ax-x^2} = xz$, eritque inito calculo $S\frac{x^2dx}{\sqrt{ax-x^2}} = S\frac{-2a^3dz}{\overline{z^2+1}^3} = 2a^3 S\frac{-dz}{\overline{z^2+1}^3}$. Juxta Canonem Corollarii IX. instituatur aequatio $S\frac{-dz}{\overline{z^2+1}^3} = \frac{Bz^3 \pm Cz^2 \pm Dz \pm K}{\overline{z^2+1}^2} \pm H S\frac{dz}{z^2+1}$, hacque de more differentiata, obtinebitur

$$-\begin{matrix}Bz^6\\ \pm Hz^6\end{matrix}\;\; 2Cz^5 - \begin{matrix}3Dz^4\\ \pm 2Bz^4\\ \pm 3Hz^4\end{matrix} - 4Kz^3 - \begin{matrix}2Dz^2\\ \pm 3Bz^2\\ \pm 3Hz^2\\ \pm z^2\end{matrix} \pm \begin{matrix}2Cz\\ -4Kz\end{matrix} \pm \begin{matrix}D\\ \pm H\\ \pm 1\end{matrix} = 0;$$

unde, singulis terminis nihilo aequatis, eruitur $H = -\frac{3}{8}$, $B = -\frac{3}{8}$, $C = 0$, $D = -\frac{5}{8}$, $K = 0$, proindeque $S\frac{-dz}{\overline{z^2+1}^3} = -\frac{\frac{3}{8}z^3 - \frac{5}{8}z}{\overline{z^2+1}^2} - \frac{3}{8}S\frac{dz}{z^2+1} \pm \phi$ &

$$S\frac{-2a^2dz}{\overline{z^2+1}^3} = -\frac{\frac{3}{4}a^2z^3-\frac{5}{4}a^2z}{\overline{z^2+1}^2} - \frac{3a^2}{4}S\frac{dz}{z^2+1} + \phi.$$

Jam vero $\frac{-3a^2}{4}S\frac{dz}{z^2+1}$, seu $-\frac{3a}{4}S\frac{a^2 \times adz}{a^2z^2+a^2} =$ $-\frac{3a}{4}$ in arcum circuli, cujus radius sit a, tangens az, seu $\frac{a}{x}\sqrt{ax-x^2}$. Hic arcus vocetur A, & loco ipsius z in æquatione superiori substituatur $\sqrt{\frac{ax-x^2}{x}}$, orietur Spatium Cissoidale $= -\frac{1}{4}\times\overline{3a+2x}\,\sqrt{ax-x^2} - \frac{3a}{4}\times A + \phi$. Ut determinetur constans $\phi$, ponatur $x = 0$; & Spatium Cissoidale evanescet, ut constat; & tangens $\frac{a\sqrt{ax-x^2}}{x}$, seu $\frac{a\sqrt{a-x}}{\sqrt{x}}$ fiet $= \frac{a\sqrt{a}}{0} = \infty$, proindeque arcus A evadet circumferentiæ quadrans, seu Q, eritque $\phi = \frac{3a}{4}\times Q$, ac denique Spatium Cissoidale prodibit $= \frac{3a}{4}\times\overline{Q-A} - \frac{1}{4}\times\overline{3a+2x}\,\sqrt{ax-x^2}$. Notum est autem, tang. $\overline{Q-A}$ esse tertiam proportionalem ad tang. A, & circuli radium; sive ad $\frac{a\sqrt{ax-x^2}}{x}$, & a, adeoque tang. $\overline{Q-A} = \frac{ax}{\sqrt{ax-x^2}}$; si ergo Q — A vocetur D, erit Spatium Cissoidale $= \frac{3a}{4}\times D - \frac{1}{4}\times\overline{3a+2x}\,\sqrt{ax-x^2}$. Ut habeatur Spatium Cissoidale integrum infinite productum, poni debet $x = a$, ut patet ex Schematis inspectione, tumque tang. D fiet $= \frac{a^2}{0} = \infty$, adeoque D = Q, & Spatium quæsitum $= \frac{3}{4}a \times Q$. Porro factum ex radio a in peripheriæ quadrantem Q aream semicirculi adæquat; area autem semicirculi, cujus radius sit a, quadrupla est areæ semicirculi radio $\frac{1}{2}$ a descripti, sive semicirculi genitoris; ergo factum $\frac{3}{4}$ a $\times$ Q triplum est semicirculi genitoris; ac proinde Spatium ipsum Cissoidale infinite productum erit ejusdem semicirculi triplum.



## COROLL. X.

Hactenus supposuimus exponentes n, m, r Formulæ $\frac{z^n dz}{z^m \pm a^{m^r}}$ esse numeros integros; at li-

licet exponentes iidem ſint fracti rationales quicumque poſitivi, & negativi, regula ſupra expoſita non deficiet, ſed ſemper e voto cedet ubi fuerit vel $r-1-\frac{n-1}{m}$, vel $\frac{n+1}{m}-1$ numerus integer ſive affirmativus, ſive negativus. Fiat enimvero $z^m \pm a^m = x z^m$, initoque calculo deprehendetur $\frac{z^n dz}{\overline{z^m \pm a^m}^{\,r}} = \frac{\ast a^{n\ast 1-mr} x^{-r} dx}{m} \times \overline{x-1}^{\,r-1-\frac{n-1}{m}}$, quæ per ſuperiores Canones integratur, ſi index $r-1-\frac{n-1}{m}$ fuerit numerus integer, ſive affirmativus, ſive negativus. Fiat inſuper $z^m \pm a^m = x$, & prodibit $\frac{z^n dz}{\overline{z^m \pm a^m}^{\,r}} = \frac{x^{-r} dx}{m} \times \overline{x \pm a^m}^{\,\frac{n+1}{m}-1}$, quæ pariter integratur ex dictis, ſi $\frac{n+1}{m}-1$ ſit numerus integer affirmativus, vel negativus.

## COROLL. XI.

Æquum nunc eſt, ut nonnulla de Fractionibus trinomialibus adjiciamus, a quibus Tayloriani Problematis ſolutio dependet. Integranda ſit igitur fractio $\frac{z^m dz}{e \pm f z^n + g z^{2n}}$, in qua m deſignet numerum integrum ſive poſitivum, ſive negativum, n numerum integrum affirmativum. Reſolvatur denominator $e \pm f z^n + g z^{2n}$, ut ſupra documus, in componentes trinomiales quadraticos $F^2$, $G^2$, $I^2$, &c. ita ut ſit $\frac{z^m dz}{e \pm f z^n \ast g z^{2n}} = \frac{z^m dz}{F^2 . G^2 . I^2 \,\&c.}$. Inſtituatur æquatio $\frac{z^m dz}{e \pm f z^n + g z^{2n}} = \frac{A z^{m+1} dz + B z^m dz}{F^2} \ast \frac{C z^{m+1} dz + D z^m dz}{G^2} \ast \frac{E z^{m+1} dz + H z^m dz}{I^2} \ast$ &c., reductiſque fractionibus ad eundem denominatorem, & comparatis ſingulis terminis inter ſe invenientur valores aſſumptarum A, B, C, D, E, H, &c quibus ſuo loco ſubſtitutis integrabuntur per ſuperiores regulas fractiones ſingulæ, quæ trinomium quadraticum in denominatore complectuntur, collectiſque ſingulis terminis obtinebitur integratio Fractionis $\frac{z^m dz}{e \pm f z^n + g z^{2n}}$.





Si

Si fractio fuerit $\frac{z^m dz}{e \pm f z^{-n} + g z^{-2n}}$, multiplicentur tum numerator, tum denominator per $z^{2n}$, & convertetur Fractio data in hanc $\frac{z^{m+2n} dz}{g \pm f z^n + e z^{2n}}$, quam modo integrare docuimus.

## COROLL. XII.

Si fractio data fuerit $\frac{z^{\frac{r}{m}} dz}{e \pm f z^n + g z^{2n}}$; fiat $z = x^m$, eritque $\frac{z^{\frac{r}{m}} dz}{e \pm f z^n + g z^{2n}} = \frac{m x^{m + r - 1} dx}{e \pm f x^{nm} + g x^{2nm}}$, quæ per regulas traditas integratur. Si denique fuerit $\frac{z^{\frac{r}{m}} dz}{e \pm f z^{\frac{n}{t}} + g z^{\frac{2n}{t}}}$ $\frac{z^{\frac{r}{m}} dz}{e \pm f z^{\frac{n}{t}} + g z^{\frac{2n}{t}}}$, fiat $z = x^{tm}$, & prodibit $\frac{z^{\frac{r}{m}} dz}{e \pm f z^{\frac{n}{t}} + g z^{\frac{2n}{t}}} = \frac{m t x^{m t \ast r t - 1} dx}{e \pm f x^{mn} + g x^{2mn}}$, quæ rursus notis regulis subjacet.

Plu-



Plura hac de re dicere supersedeo, quum præsertim hanc Spartam *De Polynomiorum Resolutione* præter eximios Geometras Joannem Bernoullium, Jacobum Hermannum, Gabrielem Manfredium, Julium Fagnanum, Jacobum Riccatum, Abrahamum De Moivre, sublimem Newtonum, magnum Eulerum, immortalem Alembertium, aliosque complures, nuper egregie adornaverit acutissimus, doctissimusque Thomas le Seur ex Minimorum Familia in elegantissimo Commentario, cui titulus *Memoire sur le Calcul Integral*, Romæ edito anno 1748; ubi nova traditur methodus perspicuitatis, & elegantiæ plenissima integrandi per Sectionum Conicarum quadraturas quantitatem $\frac{p\,dx}{q}$, designantibus p, & q polynomia quæcumque, uti $a \ast b x^m \ast c x^n \ast$ &c., vel eorundem producta, aut potestates integras; quæ omnia Cl. Auctor ea concinnitate, & elegantia ad umbilicum perducit, ut intimam penitioris Analyseos cognitionem brevi licet Commentariolo jure admireris.

H 4 OPU-

# OPUSCULUM III.

## DE INVENIENDA FORMULA RADII Osculatoris in Curvis ad umbilicum relatis ex data Formula ejusdem in Curvis relatis ad axem, eruendisque inde Curvarum Evolutis.

RAdii Osculatoris differentialis Formula pro Curvis Umbilicalibus prostat passim apud omnes fere Calculi Differentialis Scriptores, subtiliter illa quidem inventa, ingenioseque ex Curvarum hujusmodi natura deprompta; sed nemo hactenus, quod ego sciam, in animum induxit eandem eruere ex Formula altera Radii Osculatoris Curvarum ad axem relatarum; cum tamen ex hac adeo eleganter, nitideque illa deducatur, ut mirari subeat præstantissimos Analyseos Scriptores nunquam hac de re cogitasse. Id itaque mihi sumpsi, ut Formulam ipsam inde derivarem, novaque methodo generalem Formulam pro Curvarum Evolutis detegerem. Sit igitur

### PROBLEMA I.

Invenire methodum deducendi Formulam Radii Osculatoris in Curvis ad focum relatis ex data Formula ejusdem pro Curvis relatis ad axem.





### SOLUTIO

Esto (Fig. v.) Curva ABC ad focum D relata, cujus invenire oporteat Radium Osculatorem ex data Formula Radii Osculatoris Curvæ alterius ad axem relatæ. Vocetur BD z, nC dz, & arcus infinitesimus Bn centro D intervallo DB descriptus exprimatur per du. Concipiatur eadem Curva ABC relata ad quemcumque axem DF, qui transeat per focum D. Vocentur DE x, BE y, EF dx, Cm dy. Triangula rectangula BnC, BmC, BED binas suppeditant æquationes $dx^2 + dy^2 = dz^2 + du^2$, & $z^2 = x^2 + y^2$, quarum ope inveniendus est Radius Curvæ Osculator datus in z, & du.

Ut hoc obtineatur, in Formula generali Radii Osculatoris pro Curvis ad axem relatis, seu in $\frac{\overline{dx^2 + dy^2}^{\frac{3}{2}}}{dyddx - dxddy}$ subrogare oportet loco dx, dy, ddx, ddy, valores earundem ex binis æquationibus superioribus deductos. Itaque pro $\overline{dx^2 + dy^2}^{\frac{3}{2}}$ substituetur $\overline{du^2 + dz^2}^{\frac{3}{2}}$ uti ex superioribus invenitur. Porro ut eruatur valor quantitatis dyddx — dxddy, animadverto quantitatem hanc æqualem esse alteri $\frac{\overline{dyddx - dxddy} \times dy^2}{dy^2}$, quæ exprimit differentiale quantitatis $\frac{dx}{dy}$ ductum in $dy^2$; atque idcirco quæro primum valores ipsarum dx, & dy, quibus inventis protinus obtinetur valor $dyddx - dxddy$ expressus dumtaxat per z, du, & earum differentias. Fiat igitur æquationis $z^2 = x^2 + y^2$ differentiatio; & prodibit $zdz = xdx + ydy$, unde colligitur $dy = \frac{zdz - xdx}{y}$. Si valor iste substituatur in prima æquatione $dx^2 + dy^2 = dz^2 + du^2$, obtinebitur $y^2dx^2 + z^2dz^2 - 2zxdxdz + x^2dx^2 = \overline{dz^2 + du^2} \times y^2$, seu $\overline{z^2 - y^2} \times dz^2 - 2zxdxdz + \overline{y^2 + x^2} \times dx^2 = y^2du^2$; sed $z^2 - y^2 = x^2$, & $y^2 + x^2 = z^2$; ergo substitutione facta in æquatione modo inventa orietur $x^2dz^2 - 2zxdxdz + z^2dx^2 = y^2du^2$; eductisque radicibus alia prodit æquatio $xdz - zdx = \pm ydu$, ex qua deducitur $dx = \frac{xdz \mp ydu}{z}$. Quoniam vero $zdz - xdx = ydy$, subrogato in hac valore ipsius dx modo invento,

to,



to, obtinebitur $\frac{z^2dz - x^2dz \pm xydu}{z} = ydy$, in qua rursus loco $z^2 - x^2$ subrogata quantitate æquali $y^2$, eruetur postremo $\frac{ydz \pm xdu}{z} = dy$. Proinde fiet $\frac{dx}{dy} = \frac{xdz \mp ydu}{zdz \pm xdu}$. Ut jam dignoscatur, quodnam ex signis, quibus quantitates ydu, & xdu afficiuntur, sit accipiendum, superius ne, an inferius, satis erit perpendere, num in æquatione $xdz - zdx = \pm ydu$ quantitas xdz major, an minor sit altera zdx; ex hoc enim protinus apparebit signum quantitati ydu, quæ priorum differentia est, præfigendum. Ad hoc inveniendum triangula similia Brn, Crm viam sternunt. Erit itaque Br : Cr = nr : rm, & componendo Br + Cr : nr $\pm$ rm = Br : nr; proindeque nr $\pm$ Cr : Br $\pm$ rm < Br : nr, seu Cn : Bm < Br : nr; sed ob triangula similia Brn, BDE est Br : nr = BD : DE; ergo erit etiam Cn : Bm < BD : DE, seu dz : dx < z : x, adeoque xdz < zdx, & xdz − zdx = − ydu. Hinc pronum est colligere in quantitatibus ydu, & xdu signum dumtaxat inferius obtinere. Erit igitur $\frac{dx}{dy} =$

xdx

$\frac{xdz \pm ydu}{ydz - xdu}$, & differentiale ipsius $\frac{dx}{dy}$ inito calculo post terminorum reductionem invenietur $\frac{\overline{ydx - xdy} \times \overline{du^2 + dz^2} + \overline{y^2 + x^2} \times dzddu - \overline{x^2 - y^2} \times duddz}{\overline{ydz - xdu}^2}$ (A).

Jam vero ex superioribus est $dx = \frac{xdz \pm ydu}{z}$, & $dy = \frac{ydz - xdu}{z}$, adeoque $ydx - xdy = \frac{y^2 du \pm x^2 du}{z} = \frac{z^2 du}{z} = zdu$; si ergo in æquatione (A) loco $ydx - xdy$ subrogetur $zdu$, & $z^2$ pro $y^2 \pm x^2$ prodibit differentiale ipsius $\frac{dx}{dy} = \frac{zdu \times \overline{du^2 + dz^2} + z^2 dzddu - z^2 duddz}{\overline{ydz - xdu}^2}$, & multiplicando per $dy^2$, seu $\frac{\overline{ydz - xdu}^2}{z^2}$, erit differentiale ipsius $\frac{dx}{dy}$ ductum in $dy^2$, hoc est $dyddx - dxddy = \frac{du \times \overline{du^2 \pm dz^2} \pm zdzddu - zduddz}{z}$, ac denique $\frac{\overline{dx^2 + dy^2}^{\frac{3}{2}}}{dyddx - dxddy} = \frac{z \times \overline{du^2 + dz^2}^{\frac{3}{2}}}{zdzddu - zduddz + du \times \overline{dz^2 + du^2}}$



Inventa est igitur Radii Osculatoris Formula pro Curvis ad focum relatis ex data Formula ejusdem pro Curvis relatis ad axem. Q. E. I.

Determinato jam Radio Osculi curvarum Umbilicalium nova, & compendiaria methodo, præstat nunc Curvarum earundem Evolutam invenire. Sit itaque

PROBLEMA II.

Curvæ KAD ad umbilicum B relatæ Evolutam determinare.

SOLUTIO

Esto Curvæ KAD Radius Osculi AE, qui Evolutam continget in E, eritque punctum E in Evoluta HEO. Referatur & ipsa HEO ad focum B, & inclinata BE vocetur q, arcus infinitesimus GE dp. Ut hujus Curvæ eruatur æquatio, satis erit invenire relationem ipsarum GO, & GE datam per BE, seu dq, & dp per q. Exprimatur Radius Osculi AE per P, cujus valor ex Formula generali $\frac{x \times \overline{dz^2 + du^2}^{\frac{3}{2}}}{-zduddz + zdzddu + du \times \overline{dz^2 + du^2}}$ elicietur, substituendo pro du, & ddu valores earundem datos per z, dz, ddz ex æquatione Curvæ KAD; & prodibit P datum-dumtaxat in z, & conſtantibus. Ducatur ex foco B ad Radium Oſculi A E perpendicularis BF. Ob triangula ſimilia ACD, BAF

erit AD ($\sqrt{dz^2 + du^2}$): AC (du) = AB (z): AF ($\frac{zdu}{\sqrt{dz^2 + du^2}}$); & AD ($\sqrt{dz^2 + du^2}$): DC (dz) = BA (z): BF ($\frac{zdz}{\sqrt{dz^2 + du^2}}$).

Ergo FE = AE − AF = P − $\frac{zdu}{\sqrt{dz^2 + du^2}}$, voceturque N, quæ dabitur per z tantum. BF = $\frac{zdz}{\sqrt{dz^2 + du^2}}$, quæ datur rurſus per z, dicatur M. Erit iccirco BE = q = $\sqrt{N^2 + M^2}$ expreſſa pariter per z. Sit jam DO Radius Oſculi infinite propinquus. Arcus minimus EO, ut conſtat, æquatur differentiæ Radiorum DO, & AE infinite proximorum, ſive incremento ipſius AE, nimirum dP. His conſtitutis triangula ſimilia EGO, BFE ſuppeditant analogiam BE ($\sqrt{N^2 + M^2}$): BF (M) = EO (dP): EG



($\frac{MdP}{\sqrt{N^2 + M^2}}$). Inventus eſt igitur valor ipſius EG, ſeu dp datus per z, & dz, & BE, ſeu q data rurſus per z. Si jam ſubſtituatur in æquatione exprimente relationem inter dp, & z valor ipſius z modo inventus, eruetur æquatio altera, quæ conſtabit tantum ex q, & dp; atque ita obtinebitur æquatio Evolutæ HEO pro caſibus quibuſcumque. Factum eſt igitur quod petebatur. Illuſtremus Methodum exemplis aliquot.

## EXEMPLUM I.

Sit Curva KAD Spiralis Logarithmica, cujus æquatio eſt du = $\frac{adz}{b}$. Invenienda ſit ipſius Evoluta HEO. Quoniam du = $\frac{adz}{b}$, erit du² = $\frac{a^2dz^2}{b^2}$, & ddu = $\frac{addz}{b}$, quibus ſubrogatis in Formula Radii Oſculatoris

prodit $\frac{z\left(dz^2 + \frac{a^2dz^2}{b^2}\right)^{\frac{3}{2}}}{-\frac{azdzddz}{b} + \frac{azdzddz}{b} + \frac{adz}{b}\left(dz^2 + \frac{a^2dz^2}{b^2}\right)} = \frac{z\left(dz^2 + \frac{a^2dz^2}{b^2}\right)^{\frac{2}{3}}}{\frac{adz}{b}\left(dz^2 + \frac{a^2dz^2}{b^2}\right)}$

$= \frac{z}{a}\sqrt{b^2 + a^2} = P$. Igitur FE = P − $\frac{zdu}{\sqrt{dz^2 + du^2}}$ =

$\frac{z}{a}$

$\frac{z}{a}\sqrt{b^2+a^2} - \frac{azdz}{b\sqrt{dz^2+\frac{a^2dz^2}{b^2}}} =$

$\frac{z}{a}\sqrt{b^2+a^2} \quad \frac{-az}{\sqrt{b^2+a^2}} = N$; BF $= \frac{zdz}{\sqrt{dz^2+du^2}} =$

$\frac{b^2z}{\sqrt{b^2+a^2}} = M$; BE $= \sqrt{N^2+M^2} =$

$\sqrt{\frac{z^2}{a^2}\times\overline{b^2+a^2} - 2z^2 + \frac{a^2z^2}{b^2+a^2} + \frac{b^2z^2}{b^2+a^2}} =$

$\frac{bz}{a} = q; z = \frac{aq}{b}$. Est autem BE : BF = EO : EG = $dp$, seu $\frac{bz}{a} : \frac{bz}{\sqrt{b^2+a^2}} = \frac{dz}{a}\sqrt{b^2+a^2} : dz$; ergo $dp = dz = \frac{adq}{b}$. Jamvero æquatio $dp = \frac{adq}{b}$ pro Evoluta HEG est ipsa æquatio Spiralis Logarithmicæ, ut constat; igitur Evoluta Spiralis Logarithmicæ est Spiralis altera priori æqualis, & similis.

EXEMPLUM II.

Esto Curva KAD ad focum B relata, in qua intercepta LA inter contactum, & normalem ex foco in tangentem sit ubique constans, seu $= a$. Hujus Curvæ æquatio, sive relatio inter BA, & AC, erit, ut facile invenitur, $adu = dz\sqrt{z^2-a^2}$. Determinanda nunc sit per superiorem methodum hujus Curvæ Evoluta HEO.



Erit igitur $du = \frac{dz}{a}\sqrt{z^2-a^2}$, $du^2 = \frac{dz^2}{a^2}\times\overline{z^2-a^2}$, $ddu = \frac{ddz}{a}\sqrt{z^2-a^2} + \frac{zdz^2}{a\sqrt{z^2-a^2}} = \frac{ddz\times\overline{z^2-a^2}+zdz^2}{a\sqrt{z^2-a^2}}$, quibus valoribus subrogatis in Formula generali Radii Osculatoris $\frac{z\times\overline{dz^2+du^2}^{\frac{3}{2}}}{-zduddz+zdzddu+du\times\overline{dz^2+du^2}}$ post prolixum calculum, quem brevitatis gratia omitto, obtinebitur Radius Osculi $= \sqrt{z^2-a^2} = p$. Jam vero ob triangula similia ACD, BAF erit AD $\left(\sqrt{dz^2+\frac{dz^2}{a^2}\times\overline{z^2-a^2}}\right)$ : AC $\left(\frac{dz}{a}\sqrt{z^2-a^2}\right)$ = AB $(z)$ : AF $\left(\sqrt{z^2-a^2}\right) = p =$ AO. Evanescet itaque FE, sive N; & ex analogia AD : DC = BA : BF, prodibit BF, sive $M = a$; adeoque BE, sive $\sqrt{N^2+M^2} = a$. Hinc inclinata BE in Evoluta HEO constans semper erit,



&

& $= a$; proindeque Evoluta HEO circulus erit, cujus centrum B, radius intercepta constans LA Curvæ ex evolutione genitæ KAD. Oritur itaque Curva KAD ex evolutione circuli, ejuſque origo incipit in circuli peripheria, tangiturque a radio ejuſdem ad punctum originis ducto, & ſemper recedens a centro B per innumeras circum revolutiones in infinitum abit. Ceteras elegantiſſimas hujus Curvæ affectiones conſulto prætermitto, quia non eſt hic locus.

## EXEMPLUM III.

Eſto KAD, Spiralis Hyperbolica, cujus affectio primaria eſt ſubtangentem habere conſtantem, quæ præbet æquationem $z\,du = a\,dz$. Invenienda ſit ipſius Evoluta HEO. Ex æquatione Curvæ habetur $du = \frac{a\,dz}{z}$, $du^2 = \frac{a^2 dz^2}{z^2}$, $ddu = \frac{a z\,ddz - a\,dz^2}{z^2}$, quibus valoribus ſubrogatis in Formula generali Radii Oſculatoris

$$\frac{z \times (du^2 + dz^2)^{\frac{3}{2}}}{z\,dz\,ddu - z\,du\,ddz + du \times (dz^2 + du^2)}$$

obtinebitur Radius Oſculi, ſeu $P = \frac{z^3 + a^2 z}{a^3} \sqrt{z^2 + a^2}$. Eſt autem (Probl. II.) FE, ſeu $N = P - \frac{z\,du}{\sqrt{dz^2 + du^2}} = P - \frac{a z}{\sqrt{a^2 + z^2}} = \frac{z^3 + a^2 z}{a^3} \sqrt{z^2 + a^2} - \frac{a z}{\sqrt{z^2 + a^2}} = \frac{z^5 + 2a^2 z^3}{a^3 \sqrt{z^2 + a^2}}$; BF, ſeu $M = \frac{z\,dz}{\sqrt{dz^2 + du^2}} = \frac{z^2}{\sqrt{z^2 + d a^2}}$; BE, ſive $\sqrt{N^2 + M^2}$, ſeu $q = \frac{\sqrt{z^{10} + 4a^2 z^8 + 4a^4 z^6 + a^6 z^4}}{a^6 \times (z^2 + a^2)}$ (A); præterea GE, ſeu

$$dp = \frac{M\,dP}{\sqrt{N^2 + M^2}} = \frac{M \times (4z^4 dz + 5a^2 z^2 dz + a^4 dz)}{a^3 \sqrt{z^2 + a^2} \times \sqrt{N^2 + M^2}} =$$

$$\frac{4z^6 dz + 5a^2 z^4 dz + a^4 z^2 dz}{\sqrt{(z^2 + a^2) \times (z^{10} + 4a^2 z^6 + 4a^4 z^6 + a^6 z^4)}}$$ (B); ſi ergo ex æquatione (A) eruatur valor ipſius z datus in q, hicque ſubſtituatur in æquatione (B), obtinebitur æquatio, quæ exprimet relationem inter q, & dp. Enimvero æquatio (A) mutatur in hanc $z^{10} + 4a^2 z^8 + 4a^4 z^6 + a^6 z^4 - a^6 q^2 z^2 - a^8 q^2 = 0$, quæ, ut apparet, ad quintum gradum deprimitur. Invenietur itaque radix z æqualis functioni ipſius q, qua ſubrogata in

 æqua-

æquatione altera (B), habebitur æquatio ex solis q, & d p efformata, quæ iccirco exprimet naturam Evolutæ HEO Spiralis Hyperbolicæ; quod erat propositum.

*F I N I S.*



| Pag. | lin. | *Errata.* | *Corrige.* |
|---|---|---|---|
| 1 | 13 | arcns radio $t$ | arcus circuli radio 1 |
| 2 | 23 | fit X | fit x |
| | 24 | X √ | x √ |
| | 26 | X d X | x d x |
| | 27 | $\frac{dx}{x}$ | $-\frac{dx}{x}$ |
| 5 | 10 | $\frac{1-1+x}{\sqrt{1-x^2}}$ | $1-\frac{1+x}{\sqrt{1-x^2}}$ |
| | 13 | $-\frac{1}{\sin.\,\omega}+\phi$ | $=\frac{1}{\sin.\,\omega}$; ergo $S\frac{d\omega}{\sin.\,\omega.\cos.\,\omega} = L\ tang.(45°.+\frac{1}{2}\omega) - \frac{1}{\sin.\,\omega}+\phi.$ |
| 7 | 4 | $\sin.\,6^3$ | $3\sin\omega^3$ |
| 10 | | 10. 11. 12. 13. 15. c | C |
| 11 | | 4. 9. 10. 18. 19. 23. c | C |
| | 9 | $p-1$ | $\overline{p-1}$ |
| 20 | 7 | $h=m$ | $h = \overline{m}$ |
| 32 | 4 | X $1-x$ &c. | X $\overline{1-x}$ &c. |
| 36 | 4 | $S^{-xdx}$X | $= S^{-xdx}$X |
| 45 | 12 | $n-m-2$ | $\overline{n-m-2}$ |
| 46 | 2 | $n-m-2$ | $\overline{n-m-2}$ |
| 50 | 7 | ... X$^{n-2h+1}$ | X$\overline{n-2h+1}$ |
| | 8 | ... X$^{n-2h+1}$ | X$\overline{n-2h+1}$ |
| 54 | 6 | $S$ | X $S$ |
| 73 | 17 | $x^{2n}-1-$ | $x^{2n}-1=$ |
| | 18 | X &c. | &c. |
| | 19 | $x^2\,\overline{-\lambda x+1}$ | $x^2-\lambda x+1$ |
| 74 | 2 | X &c. | &c. |
| | 13 | $x^n+1=x+1$ | $x^n+1=\overline{x+1}$ |
| 75 | 22 | $\frac{xv}{2}$ | $\frac{v}{2}$ |
| 79 | 14 | $\phi$, | $x=\phi$, |
| 82 | 3 | peripheria, atque | periheria, & arcu BIRN, atque |
| 89 | 13 | $A^1$, $A^3$, $A^5$ | A1, A3, A5 |
| 56 | 16 | trimonii | trinomii |
| 97 | 8 | | |

| Pag. | lin. | *Errata.* | *Corrige.* |
| --- | --- | --- | --- |
| 97 | 8 | $\left(\frac{\surd X \ \&c.}{2}\right)$ | $\frac{\surd X \ \&c.}{2}$ |
| 100 | inter | l. 6. & 7. *adde* | Hinc cum eodem Eulero duo sequentia deducimus corollaria. |
| 100 | 14 | *& alibi* $\varpi$ | $\pi$ |
| 104 | 15 | $2 - 2 - 2$ | $2 - 2 - 2 -$ |
| | | $\frac{z = y^{\frac{r}{n}}}{}$ | $z = y^{r}$, |
| | | $\frac{e^{m r}}{z^2 - 4}$ ($1 + z$) | $\frac{2 m^{r}}{z^2 - 4^2}$ |
| 112 | 6 | $X$ | $K$ |
| | 7 | $z \, d^2 z$ | $z^2 \, d z$ |
| | 7 | $\frac{1}{8} a z^3$ | $\frac{1}{8} a z^3$ |
| | 11 | $\frac{z^2}{z^2 - 4}$ | $\frac{z^2}{z^2 - 4^2}$ |
| 111 | 12 | ${}^2z$ | $z^2$ |
| | 12 | | |
| 112 | 6 | $a\frac{K}{4} = \frac{1}{2}$ | $\frac{2K}{4} + \phi = \frac{1}{2}$ |
| | 10 | $\frac{2K}{4}$; ac | $\frac{2K}{4} = -2\frac{K}{4}$; ac |
| 113 | 6 | $\frac{B^3 \ \&c.}{\overline{z^2 + 1}}$ | $\frac{B z^3 \ \&c.}{\overline{z^2 + 1}^{\,2}}$ |
| 114 | 13 | eit que | eritque = |
| 116 | 13 | $\overline{z^m \pm a^m}$ | $\overline{z^m \pm a^m}^{\,r}$ |
| 123 | 6 | $\frac{x \, d z}{z \, d z}$ | $\frac{x \, d z}{y \, d z}$ |
| 124 | 3, & 4 | invenietur | invenietur = |
| 123 | 24 | $x X$ | $z X$ |
| 128 | 9 | evoluta H E G | evoluta H E O |





JOSEPH MARIA A S. JOANNE BAPTISTA
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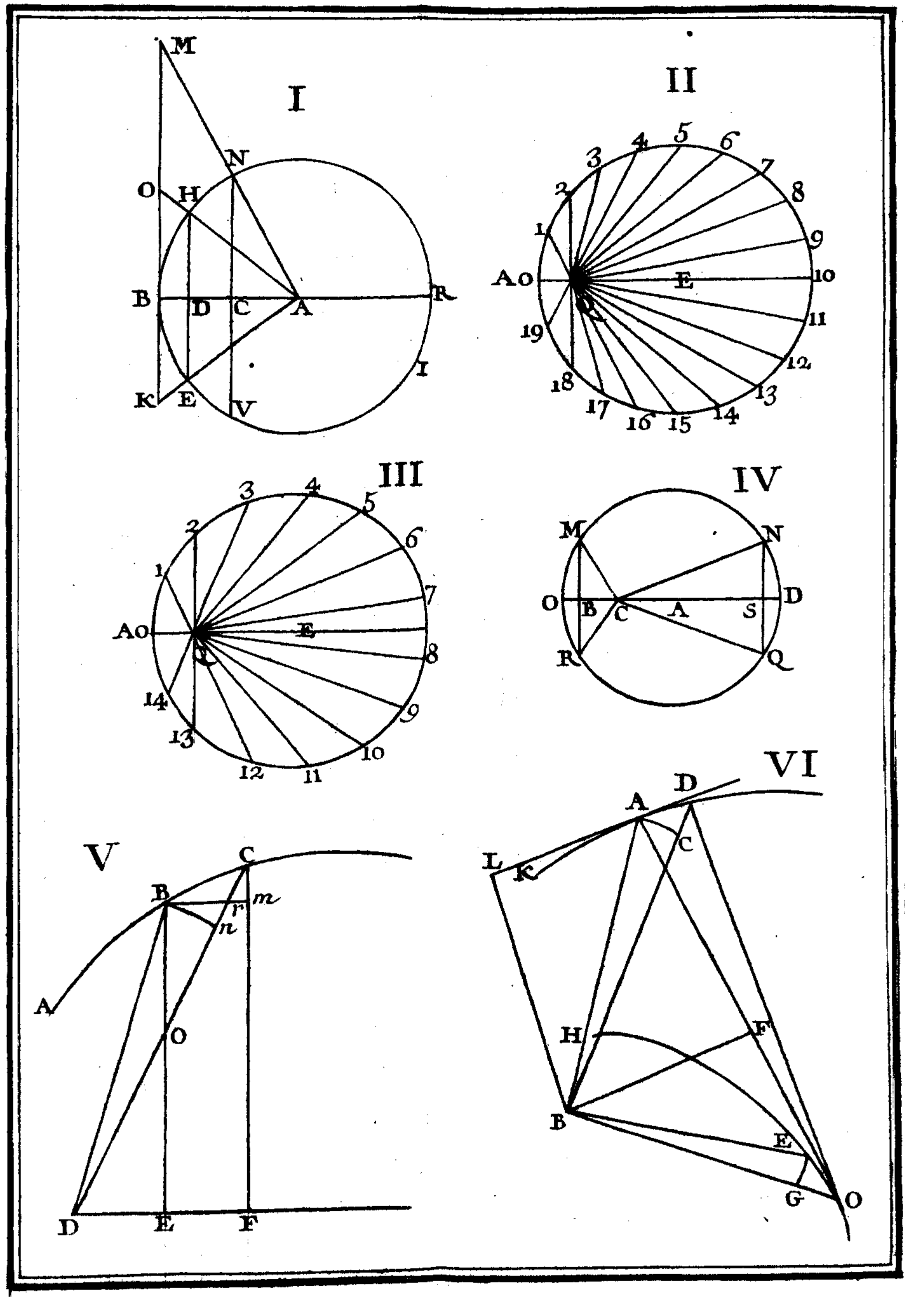

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